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LAWS of MOTION.	A THEORy of PUMPS.

To which is prefixed,

The FIRST PRINCIPLES of ALGEBRA,

By Way of INTRODUCTION.

*For the Use of the Royal Academy of Artillery at
WOOLWICH.*

VOL. I. and II.

The THIRD EDITION Improved.

With an Addition of a New TREATISE on PERSPECTIVE.

By **JOHN MULLER**, *Professor of
Artillery and Fortification.*

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AND as I ever detested Ingratitude, must address myself to You in this public Manner, knowing, your noble Disposition will consider my Inability of making any other Return; yet believe no Man can be more truly affected than myself with Your unbounded Generosity; and that I shall always be, with the greatest Sincerity,

S I R,

Your most Obliged, and

most Obedient Servant,

Royal Academy
at Woolwich,
Sept. 1764.

JOHN MULLER.

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INTRODUCTION.

§ 1. **A**LGEBRA is the Science of investigation, it proceeds by operations and rules similar to those in common arithmetic, but more general and extensive; it is grounded on the same principles as Geometry, and therefore no less clear and convincing, tho' not so natural. For in Geometry quantities are represented by their figures or images, whereas in Algebra, by symbols or letters.

§ 2. As quantity is considered to have parts and may be greater or less; it is increased by addition and decreased by subtraction, which are then the two primary operations relating to quantity. As addition and subtraction cannot be actually performed in algebra, they are represented by signs; $+$ *plus* is the mark of addition, and $-$ *minus* that of subtraction: Thus if two quantities a and b are to be added, we write $a+b$, when b is to be subtracted from a , $a-b$, and when several quantities are connected together by these signs, as $a+b-c+d$, they serve to shew which are to be added, or subtracted; thus if a is 6; b , 8; c , 7; and d , 5; then will $a+b-c+d$, be $6+8-7+5$ or 12; because 6, 8, 5 are to be added and 7 subtracted from their sum 19.

The sign $+$ or $-$, always belongs to the quantity it proceeds; thus in $a+b$, or $a-b$, the signs belong to b , and when a quantity has no sign, as a , in $a+b$, $a-b$, it is supposed to have the sign $+$, thus a , b , or $+a$, $+b$, means the same thing.

§ 3. If a quantity $a+b$ is equal to d , we write $a+b=d$; if a be greater than c , $a>c$, or if a be less than c , $a<c$; the greatest always stands at the opening of an angle.

§ 4. Quantities preceded by the sign $+$ are said to be positive, and negative when preceded by the sign $-$; thus a or $+a$, is a positive quantity, and $-b$ a negative one; they are both real, only opposite in their affections, the one increasing and the other diminishing the quantity

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ity they are joined to. Tho' $+a$, $-a$, are equal, as to quantity, yet as they are not in their affection, we do not suppose $+a = -a$, but $a - a = 0$; because to infer equality, they must not only be equal in quantity, but also in affection.

§. 5. Equal quantities are always expressed by the same letter, and unequal ones by different letters. The number of times that a quantity is taken must be joined to it; as $3a$, $5b$, means that the quantity a is to be taken 3 times, and b , 5 times: thus if a be 20 pounds, and b 10 yards; then 3 times 20 is 60 pounds, and 5 times 10, 50 yards. These numbers 3 and 5, which are joined to quantities, are called *Coefficients*.

§. 6. When several numbers, represented by letters, are to be multiplied together, they are joined as in a word, thus to multiply a by b , we write ab , and to multiply a , b , c , abc ; sometimes the sign of multiplication, $\times$ is used, especially when several quantities connected together by the signs $+$ and $-$, are to be multiplied; as when $a+b$ is to be multiplied by $c+d$, we write $\overline{a+b} \times \overline{c+d}$, with a stroke over the factors.

Observe, that one quantity cannot be multiplied by another, nor by itself, as it often has erroneously been attempted. For it would be ridiculous, to say money multiplied by money, or one debt multiplied by another: since multiplication is no more than a compendious way of adding quantities together.

§. 7. When a number is multiplied by itself, and the product by that number, the letter expressing the number is set down, with the number of factors above it, as aa , aaa , $aaaa$, is wrote a^2 , a^3 , a^4 , and these numbers 2, 3, 4, are called *indices*, or *exponents*: besides aa or a^2 , is called the square, a^3 the cube, and a^4 the fourth power of the number a .

§. 8. A quantity is said to consist of as many terms, as there are parts joined together by the sign $+$ or $-$; thus, $a+b$, consists of two terms, and is called a *binomial*, $a+b+c$ of three, and called a *trinomial*. A single quantity is that which has but one term, as ab or $+abc$.

§. 9. Quantities are said to be *like*, when they are expressed by the same letters, without considering their signs or numeral coefficients; as $+4a$ and $-3a$; or as $3abc$ and $-6abc$. These are the principal definitions necessary in algebra, the rest will be explained when used.

R E M A R K.

In the application of algebra to geometry, aa expresses a square, and a^3 a cube, whose side is a ; ab a rectangle, and abc a solid, whose sides are, a , b , c ; but we are not to conclude from thence, as we do in algebra, that the products of these sides express their contents; it is from geometry we are to know their meaning: The signs $+$ and $-$, not only express addition and subtraction in geometry, but likewise opposite affections of quantities, such as lines drawn from a point to the right and left, in a circle, the parts of a diameter taken on either side of the center, the sines above and below; in motion, contrary directions; in arithmetical questions, the sign $-$ shews whether any mistakes have been made in stating of them; for when the answer comes out negative, it shews that some error has been committed. We make this remark, to obviate the objections of unskilful authors, against some uses of the negative sign, whereby they endeavour to render algebraic computations uncertain and obscure.

A D D I T I O N.

§. 10. Is performed by connecting the several quantities together with their proper signs.

$$\begin{array}{rcl}
 \text{To } +a & +3a & 2a+3c \\
 \text{Add } +3b & -5c & +4d \\
 \hline
 \text{Sum } a+3b & 3a-5c & 2a+3c+4d
 \end{array}$$

$$\begin{array}{r}
 \text{To } 7ab-9cd+3ac \\
 \text{Add } 2bc+4ad+dd \\
 \hline
 7ab-9cd+3ac+2bc+4ad+dd.
 \end{array}$$

This is evident from the notation and explanation of the signs.

There

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There are two cases in which addition may be shortened in the following manner.

C A S E I.

§. 11. If quantities are like, and have like signs, *add their coefficients together, and prefix the sum to one of the quantities with its sign.*

To $+4a$	$-7ab$	$a+b$
Add $+3a$	$-3ab$	$3a+5b$
Sum $+7a$	$-10ab$	$4a+6b$

This rule is evident, since $4a$ and $3a$ makes $7a$, $-7ab$ and $-3ab$ makes $-10ab$; in general; whatever the coefficients of like quantities, with like signs, may be; their sum will always express the number of times that the quantity is to be taken.

To $7ab-9cd+3ac-4ad$
Add $2ab-3cd+5ac-7ad$
Sum $9ab-12cd+8ac-11ad$

C A S E II.

§. 12. If like quantities have contrary signs, *subtract the least coefficient from the greatest, and prefix the difference to one of the quantities with the sign of the greatest.*

To $+6a$	$-7b$	$-ab$	$+4a-3b$
Add $-2a$	$+3b$	$+3ab$	$-2a+4b$
Sum $+4a$	$-4b$	$+2ab$	$+2a+b$

This is evident, for, the least quantity being taken from the greatest, the difference ought to have the sign of the greatest.

To $4ab-5cd+6ac-7ad$
Add $3ab+6cd-9ac+12ad$
Sum $7ab+cd-3ac+5ad$

When several sums of like quantities are to be added together, with different signs, add all the positive terms, and all the negative ones, and then add these two sums together as before.

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	$+5a-7b$	$+6ab-7cd+8bb$
	$-3a+4b$	$-4ab-6cd+7bb$
	$-2a+3b$	$-5ab-cd-8bb$
	$+4a-5b$	$+4ab+12cd-3bb$
Sum of the positive	$+9a+7b$	$+10ab+12cd+15bb$
Sum of the negative	$-5a-12b$	$-9ab-14cd-11bb$
Total	$4a-5b$	$ab-2cd+4bb$

Observe that all like terms are to be added together, whether they stand under each other or not: for it is indifferent how they are placed, but for the sake of order, like terms are generally wrote under each other.

SUBTRACTION.

§. 13. Is performed by changing the signs of the quantities to be subtracted, then adding them as before.

From $+5a$	$-5ab$	$+6ac+3cd-5ab$
Subt. $-3a$	$+3ab$	$+3ac-cd+2ab$
Sum $+8a$	$-8ab$	$3ac+4cd-7ab$

For to subtract a negative quantity, is the same as to add its positive value; thus to subtract $-3a$, is the same thing as to add $+3a$; to subtract $+3ab$, as to add $-3ab$; and to subtract $+3ac-cd+2ab$, as to add $-3ac+cd-2ab$.

Subtraction is proved by addition as in common arithmetic: Thus $+8a$ added to $-3a$, gives $+5a$; $-8ab$ added to $+3ab$, gives $-5ab$.

From $3aa-5ac+6ab$	$5cd-4bb+7ad$
Subt. $3aa-6ac+3ab$	$3cd-6bb+4ad$
Rem. $0+ac+3ab$	$2cd+2bb+3ad$

Here $3aa$ subtracted from $3aa$, leaves nothing, since $3aa-3aa=0$.

From $13xx-15cx+7ac$	$4ab+3cd+3ac$
Subt. $12xx+cx+cd$	$3ab+5cd$
Rem. $xx-16cx+7ac-cd$	$ab-2cd+3ac$

MULTIPLICATION.

§. 14. Is performed, by multiplying all the parts of one factor, by all the parts of the other; and if the signs of the

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the terms are like, their product must be positive and negative, if the signs are unlike.

Mult. $+a$	$-2a$	$-6x$
by $+b$	$-3b$	$-4a$
Prod. $+ab$	$-6ab$	$-24ax.$

The operations of the signs is proved thus ; when a positive quantity a is multiplied by a positive number n ; it is plain that the quantity a , is to be repeated as often as n contains units, and their product must be $+na$.

If a negative quantity $-a$, be multiplied by a positive number n ; then that quantity is to be taken as often as n contains units, and the product must be $-na$, since a negative quantity, repeated ever so much, remains still negative ; to multiply a positive quantity a by a negative number $-n$, it is plain that a is to be taken so many times negatively, as n contains units ; so that the product must be $-na$.

But if a negative quantity $-a$ be multiplied by a negative number $-n$, then a is to be taken as many times with a contrary sign, as n contains units ; and to subtract a negative quantity, is the same thing as to add its positive value. Consequently, $-a$ multiplied by $-n$, gives $+na$.

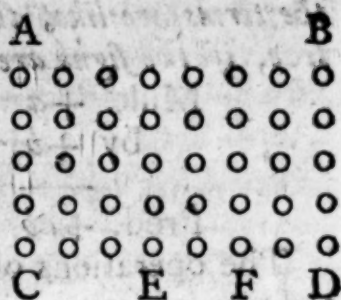
§. 15. Hence the rule, *that when the signs of the factors are both $+$ or both $-$, the product must be positive, but when the sign of one factor is $+$, and that of the other $-$, the product must be negative.*

As to the coefficients, the product of a by b is ab , (by §. 6) that of $2a$ by b , must be $2ab$, since the factor $2a$ is double the factor a ; the product of $2a$ by $3b$, must be $6ab$, because the factor $3b$ is triple the factor b , and for the same reason, the product of $4a$ by $6x$, must be $24ax$, that is 4 times the product $6ax$, of a by $6x$.

§. 16. Hence, whatever numbers the coefficients of the factors may be, *their product will be the coefficient of the product of the factors.*

It is not material in what order the letters of a product stand ; for if the marks in the line AB denote the
units

units of one factor, and the marks in the line AC, the units of the other; it is plain, that all the units in the factor AC, repeated as often as there are units in the factor AB, gives the same product as all the units in the factor AB, taken as often as there are units in the factor AC.



Hence, ab or ba , and 20×6 , or 6×20 , is the same thing, as well as abc , acb , or cad , for $ab \times c$, or $ba \times c$ and $ca \times d$ is all the same.

§. 17. The product of any two numbers AB and AC, is equal to the sum of all the products of any one of these numbers AC, and the parts CE, EF and FD of the other, divided any how. This is plain by the scheme above; that is, a multiplied by $b + c + d$ gives $ab + ca + ad$.

To pretend that a negative quantity is less than nothing, is a great absurdity, since there can be no such quantity in nature; and to say that a negative quantity is impossible in any respect, is equally absurd, for a quantity is said to be impossible, when it cannot be added to or subtracted from any other quantity; whereas it is plain, that $-a$ may be added to, or subtracted from another quantity, as well as $+a$; besides it is not the application that is here considered, but the operations.

Mult. $4a - 5c$	$7ab - 9cd + ac$
by $3b$	$4c$
$12ab - 15bc$	$28abc - 36ccd + 4acc.$

This may be proved as follows; the product of $4a$ by $3b$ is $12ab$, but as $4a$ is greater than $4a - 5c$, by $5c$, and therefore the product $12ab$ is greater than that of $3b \times 4a - 5c$, by $15bc$, which consequently must be subtracted from $12ab$, to get the required one.

N. B. In algebra, we always begin multiplication at the left, and go to the right, contrary to what is practised in arithmetic; although it is indifferent which way it is done.

Mult.

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$$\begin{array}{r}
 \text{Mult. } a+b \\
 \text{by } a+b \\
 \hline
 aa+ab \\
 +ab+bb \\
 \hline
 \text{Prod. } aa+2ab+bb
 \end{array}
 \qquad
 \begin{array}{r}
 a-b \\
 a-b \\
 \hline
 aa-ab \\
 -ab+bb \\
 \hline
 aa-2ab+bb
 \end{array}$$

For $a+b$ multiplied by a , gives $aa+ab$, and $a+b$ multiplied by b , gives $ab+bb$, therefore the sum of these two products, will be the required one; again, $a-b$ multiplied by a , gives $aa-ab$, and $a-b$ multiplied by $-b$, $-ab+bb$, by §. 15; whence the sum is $aa-2ab+bb$. If $a=6$, $b=3$, then $6+3 \times 6+3=36+2 \times 18+9=81$ and $6-3 \times 6-3=36-2 \times 18+9=9$.

$$\begin{array}{r}
 \text{Mult. } a+b \\
 \text{by } a-b \\
 \hline
 aa+ab \\
 -ab-bb \\
 \hline
 \text{Prod. } aa+0-bb
 \end{array}
 \qquad
 \begin{array}{r}
 aa+ab+bb \\
 a-b \\
 \hline
 a^2+aab+abb \\
 -aab-abb-b^2 \\
 \hline
 a^2+0+0-b^2
 \end{array}$$

For in the first example, $ab-ab=0$; in the second $aab-aab=0$, and $abb-abb=0$.

$$\begin{array}{r}
 \text{Mult. } 2a-3b-4c-6d \\
 5a-2b \\
 \hline
 10aa-15ab-20ac-30ad \\
 -4ab+6bb+8bc+12bd \\
 \hline
 10aa-19ab-20ac+6bb-30ad+8bc+12bd
 \end{array}$$

§. 18. When different powers of the same quantity are to be multiplied together, the index of the product is equal to the sum of the indices of the factors. For $aa=a^2$, $aa \times aa=a^2 \times a^2=a^4$, in general $a^m \times a^n=a^{m+n}$.

When a compound quantity is to be raised to a power, a line is drawn over it with the index at the end; thus the square of $a+b$ is marked $\overline{a+b}^2$, the cube $\overline{a+b}^3$, and the n th power $\overline{a+b}^n$.

Hence

Hence $\overline{a+b}$ multiplied by $\overline{a+b}$, gives $\overline{a+b}$; $\overline{a+b}$ multiplied by $\overline{a+b}$, gives $\overline{a+b}$; and $\overline{a+b}$ multiplied by $\overline{a+b^m}$ gives $\overline{a+b^{n+m}}$.

D I V I S I O N.

§. 19. Is performed by finding the number of times that the divisor is contained in the dividend; if both have the same sign, the quotient must be positive and negative, if they not the same.

Thus $+ab$ divided by $+b$, gives a ; $-3ab$ divided by $+3b$, gives $-2ab$; $-4abx$ divided by $-4a$, gives $+bx$ and $+21aab$ divided by $-7a$, gives $-3ab$. The proof of division is multiplication, as in arithmetic.

If the divisor is not exactly contained in the dividend; divide the part which is divisible, and set down the remainder with a line between them, as follows.

$$\begin{array}{r}
 \text{divid. } 6cd \\
 \text{by } 3b \\
 \hline
 \text{quot. } \frac{2cd}{b}
 \end{array}
 \quad
 \begin{array}{r}
 +7abc \\
 -2bd \\
 \hline
 + \frac{7ac}{2d}
 \end{array}
 \quad
 \begin{array}{r}
 +4aab \\
 +2cd \\
 \hline
 - \frac{2aab}{cd}
 \end{array}$$

For $\frac{2cd}{b}$ multiplied by b , gives $\frac{2bcd}{b}$ or $2cd$, because a quantity multiplied and divided by the same number, is neither increased or decreased; $-\frac{7ac}{2d}$ multiplied by $-2bd$, gives $+\frac{14abcd}{2d}$ or $7abc$, and $\frac{2aab}{cd}$ multiplied by $2cd$, gives $4aab$.

If the quantities are compound, set the divisor to the left of the dividend, with a stroke between them, and divide one part after another, as in arithmetic.

$$\begin{array}{r}
 \text{Divisor } 2a) \quad 6ab + 10ac \quad (3b + 5c \text{ quotient.} \\
 \underline{6ab} \quad \underline{10ac} \\
 0 \quad 0
 \end{array}$$

For

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For $6ab$ divided by $2a$, gives $3b$ for the quotient; $3b$ multiplied by the divisor $2a$, gives $6ab$, which being subtracted, leaves $10ac$; this divided by $2a$, gives $5c$ for the second part of the quotient, and the product of $5c$ by $2a$ subtracted, leaves 0. For the quotient $3b + 5c$, multiplied by the divisor $2a$, gives the dividend $6ab + 10ac$.

Divis. $a+b$ $aa+2ab+bb$ ($a+b$ Quotient.

$$\begin{array}{r} aa+ab \\ \hline ab+bb \\ \hline ab+bb \\ \hline 0 \quad 0 \end{array}$$

For aa divided by a , gives a for the quotient, and a multiplied by the divisor $a+b$, gives $aa+ab$, which taken from the dividend, leaves $ab+bb$; the first term ab divided by a , gives $+b$ for the second term of the quotient, and b multiplied by the divisor $a+b$, gives $ab+bb$, this product being taken from the remainder $ab+bb$, leaves nothing. For the divisor $a+b$, multiplied by the quotient $a+b$, gives the dividend $aa+2ab+bb$.

Divis. $a-b$ $aa-2ab+bb$ ($a+b$ Quotient.

$$\begin{array}{r} aa-ab \\ \hline -ab+bb \\ \hline -ab+bb \\ \hline 0 \quad 0 \end{array}$$

If any part of the product of the divisor, multiplied by the quotient, cannot be actually subtracted, it is set down as a part of the remainder, with a contrary sign.

Divis. $a-b$ $aa-bb$ ($a+b$ Quotient.

$$\begin{array}{r} aa-ab \\ \hline +ab-bb \\ \hline +ab-bb \\ \hline 0 \quad 0 \end{array}$$

As

As the term $-ab$ of the product cannot be subtracted, it is set down as a part of the remainder with the sign $+$. The term of the dividend should always stand according to the powers of the quantity most regularly repeated, and which is in the first place of the divisor; thus,

$$2a-x)12a^3-12aax+5axx-x^3(6aa-3ax+xx$$

$$12a^3-6aax$$

$$0 \quad -6aax+5axx$$

$$-6aax+3axx$$

$$0 \quad +2axx-x^3$$

$$2axx-x^3$$

$$0 \quad 0$$

If both the divisor and dividend are multiplied by the same quantity, strike it out and divide as usual.

Thus to divide $bca^3-10abcxx+3bcx^3$ by $abc-3bcx$, strike out bc in both, then the operation will stand thus:

$$a-3x)a^3-10axx+3x^3(aa+3ax-xx$$

$$a^3-3aax$$

$$0 \quad +3aax-10axx$$

$$+3aax-9 \quad axx$$

$$0 \quad -axx+3x^3$$

$$-axx+3x^3$$

$$0 \quad 0$$

When the divisor does not divide the dividend exactly, the remainder must be set down with the divisor under it as a part of the quotient; thus $aa+ab+bb$ divided by $a+b$, gives a and there remains bb ; and the quotient is $a + \frac{bb}{a+b}$.

In some cases it is necessary to carry on the division farther, and then the quotient becomes an infinite series, as in the annexed example.

$$a+x)$$

$$a+x) aa+xx(a-x+\frac{2x^2}{a}-\frac{2x^3}{aa}+\frac{2x^4}{a^3}-\&c.$$

$$\begin{array}{r} aa+ax \\ \hline 0. -ax+xx \\ -ax-xx \\ \hline 0 +2xx \\ +2xx+\frac{2x^3}{a} \\ \hline 0 -\frac{2x^3}{a} \\ -\frac{2x^3}{a}-\frac{2x^4}{aa} \\ \hline 0 +\frac{2x^4}{aa} \end{array}$$

The continuation of this series is so plain that it need not be carried on any farther; but x must be less than a , otherwise the series diverges instead of approaching to the value sought.

When the divisor and dividend are the same quantity raised to different powers, then the difference between the indices of the dividend and divisor, will be the index of the quotient.

Thus a^3 divided by a , gives a^{3-1} or a^2 ; a^4 divided by a^3 , gives a^{4-3} or a ; in general a^m divided by a^n gives a^{m-n} and $a+x^n$ divided by $a+x^n$, gives $a+x^{n-n}$.

The index of unity is 0; for a^n divided by a^n , gives $a^{n-n}=1$; and unity divided by a^n , gives a^{-n} ; in the same manner unity divided by $a+x^n$, gives $a+x^{-n}$.

Of the SQUARE and CUBE ROOT.

§. 20. aa is the square, and a^3 the cube of a , $aa+2ax+xx$ the square, and $a^3+3aax+3axx+x^3$ the cube of $a+x$. Hence, the square of any number divided into two parts a, x , is equal to the sum of the squares of the parts, and to twice their product; and the cube,

a

to

to the sum of the cubes of the parts, and to 3 times their product multiplied by their sum; since $3aax + 3axx = 3ax \times a + x$.

But the square root is such, as when multiplied by itself, produces the square, and the cube root, when its square multiplied by that number, produces the cube.

To extract the square root of $aa + 2ab + bb$, find the root of the first term aa , which is a , whose square being subtracted, leaves $2ab + bb$; to find the second term of the root, divide $2ab$ by twice a , the part already found, which gives $+b$, for that term; this added to the divisor $2a$, and the sum multiplied by that term, gives $2ab + bb$, which subtracted from the remainder $2ab + bb$, leaves nothing; hence $a + b$ is the root required; for $a + b \times a + b$ gives the square.

Operation $aa + 2ab + bb$ ($a + b$ root.

$$\begin{array}{r} aa \\ \hline 0 \\ 2a)2ab+bb \\ \underline{2ab+bb} \\ 0 \quad 0 \end{array}$$

If the square, whose root is sought, be $aa - 2ab + bb$; it will be found in the same manner, to be $a - b$, for that number multiplied by itself, gives the square. If a remainder be left, you continue in the same manner to divide always by twice the number already found, adding the last figure of the quotient to the divisor, and multiplying the sum by that last figure, till you get the square root exactly, if the square is compleat; thus if $aa + 2ab + bb + 2ac + 2bc + cc$, be the square, after having found the root $a + b$, of the part $aa + 2ab + bb$, divide the remainder $2ac + 2bc + cc$, by twice the root $a + b$; which gives c for the quotient; and $2a + 2b + c$, multiplied by c , gives $2ac + 2bc + cc$, which taken from the remainder, leaves nothing.

When the square is incompleat, the operation must be continued as before, till the root is found nearly.

Thus,

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Thus, to find the square root of $1 - xx$, the operations are as follows.

$$\begin{array}{r}
 1 - xx (1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{1}{128}x^8 - \&c. \\
 2) - xx \\
 \quad - xx + \frac{1}{4}x^4 \\
 \hline
 2 - x^2) \quad - \frac{1}{4}x^4 \\
 \quad - \frac{1}{4}x^4 + \frac{1}{8}x^6 + \frac{1}{64}x^8 \\
 \hline
 2 - x^2) \quad + \frac{1}{8}x^6 - \frac{1}{64}x^8 \\
 \quad - \frac{1}{8}x^6 + \frac{1}{128}x^8 \\
 \hline
 2) \quad - \frac{5}{128}x^8.
 \end{array}$$

Observe when the product cannot be subtracted, to place it to the remainder with a contrary sign, as in division; after the two first terms of the root have been found, no more of the divisor has been set down, than was sufficient to find five terms of the root only; this is sufficient to discover the law of continuation. For the numeral numerators, are 1, 1, 1, 3, 5, 7, 9, &c. and the denominators, are found, if the numerator of the last coefficient, be multiplied by the number of the next, and its denominator, by the corresponding number of the series, 1, 2, 4, 6, 8, 10, 12. Thus the coefficient $\frac{1}{8}$ of the third term, multiplied by 3, and divided by 6, gives $\frac{1}{16}$ for the coefficient of the next; this multiplied by 5, and divided by 8, give $\frac{5}{128}$, and so on.

The square root of any number, is found after the same manner; but the squares or cubes of the first 10 figures, must be previously known, for which we shall set them down here.

1	2	3	4	5	6	7	8	9	10	Roots
1	4	9	16	25	36	49	64	81	100	Squares
1	8	27	64	125	216	343	512	729	1000	Cubes

As the square of 1 the least figure is one figure, of the greatest 9 two; of the two least 10 three, of the two greatest 99 four; of the three least 100 five; of the three greatest six, and so on; *if the figures of a square number*

are divided two by two, from the right to the left; the root will consist of as many figures as there are divisions.

The square root of 321489 is found by separating them two by two as in the margin; now the nearest square less than 32 is 25, set its root 5 to the quotient, and subtract 25 from 32; to the remainder 7, join the figures 14 of the next division, divide 71 by 10, twice the first figure 5, which gives 6 for the second figure of the root or quotient, multiply that figure 6, and the divisor 10, that is, 106 by that figure 6, and subtract the product 636, from 714, which leaves 78, to this join the two next figures 89; divide 788 by 112, twice the quotient 56, which gives 7 for the third figure of the root; multiply that figure 7, and the divisor 112, that is, 1127 by 7, and subtract the product 7889, from the last remainder 7889, which leaves 0; hence 567 is the root required. For this number multiplied by itself, gives the square proposed.

$$\begin{array}{r} 32.14.89(567 \\ \underline{25} \\ 10)714 \\ \underline{636} \\ 112)7889 \\ \underline{7889} \\ \dots \end{array}$$

If the root cannot be found exactly, annex twice as many cyphers to the remainder, as you want decimals in the root and proceed as before.

To find the square root of 12 to three decimals, annex six cyphers, as here in the margin; then the nearest square less than 12 is 9, set the root 3 in the quotient, and subtract the square 9, from 12; to the remainder 3 join 00; divide 30 by 6 twice 3, which gives 4 for the second figure of the quotient, multiply this figure, and the divisor 6, or 64 by that figure 4, and subtract the product 256 from 300, which leaves 44; to this join the two next figures 00; divide 440 by 68, twice the quotient 34, which gives 6 for the third figure of the quotient; multiply this figure and the divisor 68, or 686 by 6, and subtract the product 4116 from the dividend 4400, which leaves 284; to these join the two last

$$\begin{array}{r} 12.00.00.00(3.464 \\ \underline{9} \\ 6)3.00 \\ \underline{256} \\ 68)4400 \\ \underline{4116} \\ 692)28400 \\ \underline{27696} \\ \dots 704 \text{ remainder} \end{array}$$

last figures 00, divide by 692, twice the quotient 346, which gives 4 for the last figure of the quotient: multiply this figure and the divisor, or 6924 by 4, and subtract the product 27696 from the dividend 28400, which leaves 704. Hence 3.464 is the root required; for that number squared, and the remainder 704 added, gives the square proposed.

If the root be required to many places; *find half the number as usual, and divide half the remainder by the part of the root already found, and you will get as many more.*

If the square root of 126 be required to thirteen places, find the first seven, 11.22497 as usual, and divide 2424955, half the remainder, by 11.22497, which gives 216032, and this added to the former part, gives 11.22497216032 for the root required true in all its places.

§. 21. The method of extracting the cube root, is derived from this general form, $a^3 + 3aab + 3abb + b^3$, or cube of $a+b$, for a is the root of the first term a^3 , and its cube taken from the expression, leaves $3aab + 3abb + b^3$; here it is plain, that if we divide $3aab$, by 3 times the square of the first part a , we get the second b , and b multiplied by the divisor $3aa$, gives $+3aab$; and if to $3a$, three times the first, we add the second b , and the sum $3a+b$ be multiplied by bb the square of the second, we get $3abb + b^3$, this added to the product $3aab$, and the sum subtracted from the remainder leaves nothing. The operation stands thus,

Cube $a^3 + 3aab + 3abb + b^3$ ($a+b$ root.

$$a^3 + 3aa \times b + 3a + b \times bb$$

If the root consists of more than two parts or places, the same operation which serves to find the second, serves also to find any other; that is, *three times the square of the part found already, gives the divisor; the quotient multiplied by the divisor, the first product; then 3 times the first figures, added to the last, multiplied by the square of the last, the second; and their sum subtracted from the remainder, leaves nothing if the cube be complete.*

Before this rule is applied to numbers, observe, that

the cube of the least number is one figure; of the greatest 9 three, of the two least 10 four, of the two greatest 99 six, of the three least seven, of the three greatest nine; if therefore *the figures of the cube be separated three by three from the right to the left, the root will consist of as many figures as there are divisions.*

The cube 42875, being divided into two parts as in the margin; then the nearest cube less than 42 is 27, whose root 3 is the first part of the root, and 27 subtracted, leaves 15.875: divide 158 by 27 three times the square of 3, which gives 5 for the second part, and 5 times the divisor 27 gives 135, set this product under the dividend 158; then 9 three times the first figure 3, joined to the last, and 95 multiplied by 25 the square of the last 5, gives 2375, which place so as the last figure be under the last of the remainder; then add these two products, and subtract their sum 15875 from the remainder, which leaving nothing, shews that 35 is the root required.

N. B. The last figure 5, is always joined to three times 9 of the first, because the first stands in the place of tens in respect to the last; and therefore is the same as adding 5 to 90: the same thing is true in regard to the two products of 135 which in reality is 13500, the cyphers are omitted, and 2375.

The cube 2460375 being divided into three parts; then the nearest cube less than 2 is unity, which subtracted from 2 leaves 1, to which join the next division 460, and divide 14 by 3 three times the square of the first part 1 of the root, which gives 3 for the second, and the product of the divisor 3 multiplied by the quotient 3, gives 9; then 3 three times

$$\begin{array}{r}
 42.875 \text{ (35 Root.} \\
 \underline{27} \\
 27)15.875 \\
 \underline{135} \\
 2375 \\
 \underline{15875} \\
 \dots
 \end{array}$$

$$\begin{array}{r}
 2.460.375 \text{ (135 root.} \\
 \underline{} \\
 3)1.460 \qquad \qquad 33 \\
 \underline{9} \qquad \qquad \qquad \underline{9} \\
 297 \qquad \qquad \qquad 297 \\
 \underline{} \\
 1197 \qquad \qquad \qquad \underline{} \\
 \underline{} \\
 507)263.375 \qquad \qquad 395 \\
 \underline{2535} \qquad \qquad \underline{25} \\
 9875 \qquad \qquad \underline{} \\
 \underline{} \qquad \qquad 1975 \\
 263375 \qquad \qquad \underline{790} \\
 \underline{} \qquad \qquad \underline{} \\
 \dots \dots \dots 9875 \\
 \qquad \qquad \qquad \text{the}
 \end{array}$$

the first figure joined to the second 3, and the sum 33 multiplied by 9 the square of the second, gives 297, which added as observed above, and the sum 1197 subtracted from 1460 leaves 263; to this join the last division 375; then 3 times the square of 13 the figures already found, gives 507, for the divisor, which is contained 5 times in 2633, the product of the divisor 507, and this figure 5, gives 2535: then three times the first figures 13 joined to the last 5, and the sum 395 multiplied by 25 the square of the last gives 9875, these two products added as observed above, and their sum taken from the remainder 263375 leaves nothing. Hence 135 is the root required.

In the same manner the cube root of any number may be found; but if it be required to a great many places, *Find half the number as before and divide one third of the remainder by the square of the root found, and you will get as many more places.*

Thus if the cube root of 20 be required to eight places, the four first 2.714 being found as before, then one third part of the remainder 9229656, divided by 736596 the square of 2.714, gives 4176 more, and hence 2.7144176 will be the root to eight places.

General Method for proving Arithmetical Operations.

§. 22. Any number standing in any place, being divided by 9, leaves always a remainder equal to itself; for 20, 300, 50000, divided by 9, leave 2, 3, 5; therefore any number 86754, divided by 9, leaves the same remainder, as would be left, by taking as many 9's as can be done from their single values, for $8+6+7+5+4=30$ or 3, the same as 86754 divided by 9.

Hence if $9a+r$, $9b+s$, represent any two numbers, and r, s , the remainders less than 9, it is plain that the remainders $r+s$, or $r-s$, of their sum or difference, is the same, provided 9 is taken from $r+s$, if greater than 9, as the sum or difference of the remainders, taken from these numbers separately.

In the product $81ab + 9ar + 9bs + rs$; of these numbers; the remainder rs is equal to the product of the remainders of the factors. If $81ab + 9ar + 9bs + rs$, be divided by $9a + r$, the product rs of the remainders of the divisor and quotient, its equal to the remainder of the dividend.

If $9a + r$ be the root of any power, the remainder r raised to that power, is equal to the remainder of that power, if less than 9, or if greater, its difference.

If the division or root, is not exact; the remainder left after the division or root, must be subtracted from the dividend or power, before the remainders are taken. Thus 456 divided by 12 gives 37 and leaves 12; which 12 taken from the dividend 456; the difference 444, gives 3 for the remainder, and the remainders 3, 1, of the divisor 12 and quotient 37, gives likewise 3 for the remainder.

The square root of 3136 is 56 exactly, whose remainder 2 being squared gives the same remainder 4, as that of the square. And the cube root of 2744 is 14 exactly, whose remainder 5 raised to the cube gives 125 or 8 the same as the remainder of the cube. The square root of 20 is 4.47 and leaves 191, which subtracted from the square 200, leaves 9 for the remainder, and the square of the remainder 6 of the root 4.47, is 36 or 9 as that before.

This method has been objected against, because if by mistake a 9 is wrote for a 0, it does not discover the error; which is no more than may happen in all other operations, where one error may counterballance another. Besides the proof in the square and cube root containing a great number of places, is more troublesome in the common way and more liable to errors, than finding the root itself; whereas this method may be used without a pen.

VULGAR FRACTIONS.

§. 23. A fraction consists of two parts, the one being placed

placed over the other with a stroke between them, as $\frac{2}{3}$, $\frac{b}{a}$, or $\frac{a+b}{c}$.

The under one shews in how many equal parts the quantity is divided and is called the *Denominator*, and the upper, how many of these parts the fraction contains and is called the *Numerator*. Thus in the fraction $\frac{3}{4}$, the denominator 4 shews that the quantity is divided into 4 and the numerator 3, that this fraction contains 3 of these parts. If $\frac{3}{4}$ is a fraction of a pound or 20 shillings it will be 15 shillings, because the fourth part of 20 is 5, and 3 times 5 is 15.

When the numerator is greater than the denominator, as in $\frac{4}{3}$ or $\frac{7}{4}$, the fraction is said to be *improper*; hence any whole number may be said to be an improper fraction whose denominator is unity: $\frac{4}{1}$ of a fathom or 6 feet is 8 feet; because one third of 6 is 2, and twice 4 is 8: Again, $\frac{7}{5}$ of a pound or 20 shillings is 28 shillings, for the fifth part of 20 is 4, and 4 times 7 is 28.

To reduce a FRACTION to its lowest Denomination.

§. 24. Divide, if possible, both numerator and denominator, by their greatest common divisor, and the quotients will be the fraction required; but if they have no common divisor, the fraction cannot be reduced.

The fraction $\frac{6}{12}$ divided by 6, gives $\frac{1}{2}$; $\frac{ab}{ac}$ divided by a gives $\frac{b}{c}$; $\frac{56}{240}$ divided by 8 gives $\frac{7}{30}$; $\frac{ab+ac}{ac}$ divided by a gives $\frac{b+c}{c}$.

This rule is evident, because a quantity multiplied and divided by the same number, neither increases nor diminishes its value: for $\frac{na}{nb} = \frac{a}{b}$, $\frac{10}{20} = \frac{1}{2}$.

Fractions should always be reduced to their lowest denomination before any other operation is performed, because they become more easy and less perplexing; which may be done in many cases as follows.

§. 25. All

§. 25. All numbers ending with a 5 or 0, are divisible by 5; all numbers whose sums make any number of 9's exactly, when their single values are only considered, are divisible by 9; all numbers whose sums make any number of 6's are divisible by 6, when they end with even numbers, otherwise by 3.

The fraction $\frac{120}{345}$ being divisible by 5 gives $\frac{24}{69}$; the fraction $\frac{567}{81}$ being divisible by 9 gives $\frac{63}{9}$ and by 9 again gives $\frac{7}{1}$; the fraction $\frac{272}{486}$ being divisible by 6 gives $\frac{62}{81}$, and the fraction $\frac{372}{493}$, by 3 only gives $\frac{124}{163}$.

§. 26. *To find the greatest common divisor of any two numbers.*

Divide the greatest by the least; divide the least by the remainder, continue always to divide the last divisor by the last remainder, till you find a number that divides exactly, which will be the divisor sought.

To find the greatest common divisor of 2346 and 89046; divide the last by the first; the divisor 2346 by the remainder 2244, and this last by the remainder 102, which leaving 0, will be the divisor sought. For 2346 divided by 102, gives 23, and 89046 by 102 gives 873.

The greatest common divisor of algebraic expressions is found in the same Manner, only observing, *always to divide the expressions by their simple divisors if they have any, and then proceed as before.*

Thus in $ac+cx$, and $aax+2axx+x^3$, divide the first by c , and the second by x ; then $a+x$, and $aa+2ax+xx$, by $a+x$, which dividing exactly will be the divisor sought.

If both expressions have a simple common divisor, divide them by it; and if they have also a compound common divisor; *then the product of these two common divisors will be the required one.*

Thus $aax-xx^3$ and $aax-2axx+x^3$ being divided by x , and the quotients $aa-xx$, $aa-2ax+xx$, by $a-x$, which dividing exactly gives $xa-x$ or $ax-xx$ for the divisor required.

To prove this rule, let m and n be any two numbers,

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bers, whose common divisor is required; if m divided by n quotes a and leaves p ; n divided by p quotes b and leaves q ; and p divided by q quotes c without a remainder: then as the product of the quotient multiplied by the divisor, added to the remainder is equal to the dividend, we have $m = na + p$, $n = pb + q$, $p = cq$; and as q divides p exactly, by supposition, it will also divide its multiple pb , and itself; and as q divides $pb + q$ it will divide its equal n , or any multiple na of n , it will then divide $na + p$ or its equal m ; and therefore q divides m and n exactly, and since q is the first common divisor found, it must be the greatest that these numbers can have.

When the ratio of two numbers cannot be reduced by a common divisor, it is often useful to do it by approximation, as follows.

If the remainder p be neglected, then $m = an$, or the ratio of m to n is that of a to unity; if the remainder q be neglected, then $n = pb$, and this value of n wrote into $m = na + p$ gives $m = abp + p$, and m is to n as $abp + p$, to bp as $ab + 1$ to $b + 0$: If the remainder in the next division be neglected; then $p = cq$; this value of p wrote into $n = bp + q$ gives $n = bcq + q$, and these of n and p into $m = na + p$, gives $m = abcq + aq + cq$: Hence m is to n as $abcq + aq + cq$, to $bcq + q$, or as $abc + a + c$ to $bc + 1$. The approximation may be continued by this

GENERAL RULE.

§. 27. Set down the ratio of unity to 0, and under it that of the first quotient to unity; then to find the succeeding ones, multiply always the last ratio by the next quotient, and add to it the next ratio, antecedent to antecedent, and consequent to consequent.

N. B. The first, third, fifth, seventh ratios, are always greater, and the second, fourth, sixth, &c. always less than the true one.

Example. The diameter of a circle being unity, the circumference will be 3.14159, this last number divided

vided by 1, gives 3, and there remains .14159, with this divide unity or 1.00000, which gives 7, and the remainder is 887, with this divide 14159, and the quotient will be 15, and dividing the last divisor by the last remainder, we get unity for the quotient. Now since 3, 7, 15, 1, are the quotients we get by the rule. Therefore the ratio of the diameter to its circumference is greater than that of 7 to 22, less than that of 106 to 333, and greater than that of 113 to 355.

$$\begin{array}{rcl}
 < 1 & : & 0 \\
 > 3 & : & 1 \\
 < 22 & : & 7 \\
 > 333 & : & 106 \\
 < 355 & : & 113
 \end{array}$$

To find the least number divisible by any number of Figures.

§. 28. Divide all the numbers that can be by the same divisor, set down the quotients and the remaining numbers, which divide likewise by some common divisor; continue so till there remains no numbers that have a common divisor; then the product of all the divisors, and the last remaining numbers, will be the required one.

Let $a, 3a, b, 2b, 3c, d$: express the given numbers.

Then a) 1, 3, $b, 2b, 3c, d$,

b) 1, 3, 1, 2, $3c, d$,

3) 1, 1, 1, 2, c, d ,

Hence $2 \times 3 \times abcd$, or $6abcd$, is the number sought. For 6 is divisible by 2 and 3, and $abcd$, by a, b, c, d . It is plain that this number is the least divisible by the given number.

To reduce FRACTIONS under the same Denomination.

§. 29. Multiply the several numerators separately, by all the denominators, excepting its own; then the products will be the numerators, and the product of all the denominators, the common denominator.

The fractions $\frac{2}{3}, \frac{5}{7}$, are reduced under the same denomination, if we multiply the numerator 2, by the denominator 7, and the numerator 5, by the denominator 3, which gives 14, 15 for the numerators, and the pro-

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product 21 of the denominators 3, 7 the common denominator. Hence $\frac{2}{3} = \frac{14}{21}$ and $\frac{3}{7} = \frac{9}{21}$.

The fractions $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, are respectively equal to $\frac{adf}{bdf}$, $\frac{bcf}{bdf}$, $\frac{bde}{bdf}$.

This rule is evident, since we do no more than to multiply the numerator and denominator by the same number, which neither increases nor diminishes its value. To reduce the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, under the same denomination, multiply the first numerator 1 by 3×4 ; the second 2, by 2×4 , and the last 3, by 2×3 , which gives 12, 16, 18, for the numerators, and $2 \times 3 \times 4$, or 24 for the common denominator. Hence $\frac{1}{2} = \frac{12}{24}$, $\frac{2}{3} = \frac{16}{24}$, and $\frac{3}{4} = \frac{18}{24}$.

The operation may be shortened when some of the denominators have a common divisor, by finding the least number divisible by all the denominators; for *that number divided by the denominator of each fraction, and the quotient multiplied by the numerator, the product will be the numerator, and the first number the denominator.*

In the fraction $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{5}{16}$, where 16 is the least number divisible by all the denominators, we get $\frac{1}{2} = \frac{8}{16}$, $\frac{3}{4} = \frac{12}{16}$, $\frac{1}{8} = \frac{2}{16}$ and $\frac{5}{16}$: If the fractions are $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{6}$, $\frac{3}{10}$, where 60 is the least number divisible by all the denominators, we get $\frac{1}{3} = \frac{20}{60}$, $\frac{2}{5} = \frac{24}{60}$, $\frac{1}{6} = \frac{10}{60}$, and $\frac{3}{10} = \frac{18}{60}$.

To add and subtract FRACTIONS.

§. 30. Reduce them under the same denomination, then the sum or difference of the numerators divided by the common denominator, will be the fraction required.

For $\frac{2}{3} + \frac{3}{4}$ gives $\frac{8}{12} + \frac{9}{12} = \frac{17}{12}$; $4 - \frac{2}{3}$ gives $\frac{12}{3} - \frac{2}{3} = \frac{10}{3}$; $\frac{1}{2} + \frac{3}{4} - \frac{1}{8}$, gives $\frac{4}{8} + \frac{6}{8} - \frac{1}{8} = \frac{9}{8}$. Observe that the denominator of any whole number is always unity.

This rule is evident, since like parts may be added or subtracted in the same manner as whole numbers; that is, thirds to thirds, fourths to fourths, &c.

To

To multiply or divide FRACTIONS.

§. 31. In multiplication, the product of all the numerators, divided by the product of all the denominators, gives the product: In division, invert the fraction of the divisor, then the division becomes a multiplication.

Thus $\frac{1}{2}$ multiplied by $\frac{3}{4}$, gives $\frac{3}{8}$; 20 multiplied by $\frac{3}{7}$, gives $\frac{60}{7}$; a multiplied by $\frac{b}{c}$ gives $\frac{ab}{c}$, and $\frac{b}{a}$ multiplied by $\frac{d}{c}$ gives $\frac{bd}{ac}$: $\frac{3}{4}$ divided by $\frac{2}{5}$ gives $\frac{3}{4} \times \frac{5}{2}$ or $\frac{15}{8}$: $\frac{5}{6}$ divided by $\frac{7}{8}$ gives $\frac{5}{6} \times \frac{8}{7}$ or $\frac{40}{42}$: a divided by $\frac{c}{d}$, gives $a \times \frac{d}{c}$ or $\frac{ad}{c}$.

To prove this rule, let $\frac{b}{a} = m$, $\frac{d}{c} = n$; then mn will be equal to $\frac{b}{a} \times \frac{d}{c}$; now as the divisor multiplied by the quotient, is equal to the dividend, we have $b = am$, and $d = cn$; but since equals multiplied by equals, produce equals, $bd = acmn$; and as equals divided by equals, gives equal quotients if we divide by ac , we get $\frac{bd}{ac} = mn$.

If any numerators are divisible by the same number, as the denominators, divide as many in the one as in the other, and multiply the quotients as before.

Thus $\frac{3}{4} \times \frac{2}{3} = \frac{1}{2} \times 1 = \frac{1}{2}$; $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{5}{6}$ gives $1 \times 1 \times \frac{1}{4} \times \frac{5}{6} = \frac{5}{24}$: $\frac{2}{3} \times \frac{6}{8} \times \frac{4}{3} \times \frac{5}{6} = 1 \times \frac{1}{2} \times 4 \times \frac{1}{6} = \frac{1}{3}$. For $\frac{b}{a} \times \frac{a}{c} \times \frac{c}{d}$, gives $\frac{abc}{acd}$ by the common rule, and $\frac{b}{d}$ when reduced.

The square or cube root of any fraction, is found by extracting the root of both the numerator and denominator separately: thus the square root of $\frac{25}{49}$ is $\frac{5}{7}$, the cube root of $\frac{125}{343}$ is $\frac{5}{7}$.

To reduce a quantity into a fraction of a higher denomination;

nomination; divide the quantity by the number of times, that the lower denomination is contained in the bigger.

Pence are reduced into a fraction of a shilling or pound, if they are divided by 12 or 240, the number of pence in a shilling or pound; shillings into a fraction of a pound, if divided by 20; inches are reduced into a fraction of a foot or yard, if they are divided by 12 or 36, the number of inches in a foot or yard; thus 3 pence gives $\frac{3}{12}$ or $\frac{1}{4}$ of a shilling, or $\frac{3}{240} = \frac{1}{80}$ of a pound.

To reduce fractions into whole numbers of a lower denomination; multiply the numerator by the number of times that its denomination is contained in the bigger.

The fraction $\frac{1}{8}$ of a pound, multiplied by 20, gives $2 + \frac{4}{20}$ shillings and $\frac{4}{20}$, or $\frac{1}{5}$ multiplied by 12, gives $2 + \frac{2}{3}$ pence; the fraction $\frac{1}{3}$ of a pound, multiplied by 20, gives 16 shillings; $\frac{1}{6}$ of a foot multiplied by 12, gives 10 inches.

To reduce a fraction into decimals, annex as many cyphers to the numerator as you want decimals, and divide by the denominator.

Thus $\frac{7}{8}$ is reduced into decimals, by annexing three cyphers to the numerator 7, and divide 7000 by 8, which gives .875; $\frac{1}{32}$ by joining 5 cyphers, and dividing by 32, gives .03125; $\frac{1}{12}$ gives .4166, and there remains $\frac{2}{3}$.

Decimals are added and subtracted like whole numbers, observing only to place the points, which separate the wholes from the decimals, under each other. They are multiplied and divided as integers, by observing to make as many figures decimals in the product, as are in both factors; and in the quotient, as there are more in the dividend than in the divisor.

Thus .125 multiplied by 6.4, gives .8000, or .8 only: .025 multiplied by .3, gives .0075; .465 divided by .5, gives .93; and .378 divided by 63, gives .006.

Decimals are reduced into common fractions, if divided by unity, with as many cyphers added to it, as there are

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are decimals; the fraction 1.5 gives $1\frac{1}{2}$ or $\frac{3}{2}$, the fraction .875 gives $\frac{875}{1000}$, or $\frac{7}{8}$ when reduced.

The value of a decimal fraction is found if multiplied by the number of times, that the lower denomination is contained in the higher, and pointing off as many decimals as there were before.

The decimals .875 of a pound, multiplied by 20, gives 17.500, and .500, or .5 multiplied by 12 gives 6 pence: .03125 of a pound, multiplied by 20, gives .625, of a shilling; this multiplied by 12 gives 7.5 pence, and 5 multiplied by 4 gives 2 farthings. The fraction of a fraction is found, by multiplying them as usual: $\frac{2}{3}$ of $\frac{3}{4}$ of 20 shillings, is $\frac{2}{3} \times \frac{3}{4} \times 20 = \frac{120}{12}$ or 10 shillings; and $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{1}{4}$ of 48 shillings is $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times 48 = 2$ shillings, for one fourth of 48 is 12, one third of 12 is 4, and one half of 4 is 2.

The RULE of THREE or PROPORTION.

§. 32. Quantities of the same kind may be compared together two ways; that is, by excess or contents, the first is called *Arithmetical*, and the second *Geometrical*.

The first term of the comparison is called the *Antecedent*, and the second the *Consequent*.

§. 33. An arithmetical progression, is a series of terms, which exceed each other by the same difference, as 2, 4, 6, 8, or $a, a+x, a+2x, a+3x, a+4x$; and hence the sum $2a+4x$ of the first a , and last $a+4x$, is always equal to the sum of any two equidistant ones, as $a+x, a+3x$, or double the middle term, when they are an uneven number.

§. 34. The sum of all the terms of an arithmetical progression, is equal to the sum of the first and last multiplied by half their numbers.

The sum of the first and last, being always equal to the sum of any two equidistant ones, the total sum is equal to the sum of the first and last multiplied by half their number. Thus if a be the first, b the last, and n their number, then $\overline{a+b} \times \frac{1}{2}n$ will be the sum.

Ex.

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Ex. The sum of 20, terms of the natural numbers 1, 2, 3, 4, &c. is equal to $1 + 20 \times \frac{1}{2} \times 20 = 210$:

The sum of 41 terms of the series 3, 5, 7, 9, 10, &c. is $3 + 83 \times \frac{1}{2} \times 41 = 1763$.

§. 35. If of four quantities, the quotient of the first and second, be equal to the quotient of the third and fourth, those quantities are said to be in a geometrical proportion, such are the numbers 2, 4, 6, 12; or a, ra, b, rb ; the geometrical proportion is wrote thus, $a : ra :: b : rb$; which means, that a is to ra as b is to rb .

Hence if four quantities 2, 4, 6, 12, or a, ra, b, rb , are proportional, the product of the extremes is equal to that of the means. For $2 \times 12 = 4 \times 6$, or $a \times rb = ra \times b$: and on the contrary, if four quantities are such, that the product of the means is equal to that of the extremes, they are proportional.

§. 36. If the two middle terms of a proportion are equal as in $a : ar :: ar : arr$, the proportion is said to be continued, or in $4 : 6 :: 6 : 9$. Hence,

1°. To find a fourth proportional to three given numbers a, ar, b , or 4, 6, 8: multiply the second by the third, and divide by the first, then the quotient br or 12 will be the number required. Since $a : ar :: b : br$, or $4 : 6 :: 8 : 12$; §. 35.

2°. To find a third continued proportional to two given numbers, a, ar , or 4, 6, divide the square of the second by the first, and the quotient arr or 9 will be the number sought: Since $a : ar :: ar : arr$, or $4 : 6 :: 6 : 9$.

3°. To find a mean proportional, between two given numbers, a, arr , or 4, 9. The square root of the product of these two numbers, gives the mean ar , or 6, sought.

§. 37. If several quantities are in a continued proportion, as a, ar, arr, ar^2 , or 2, 4, 8, 16. Then 1°. The square of the first is to the square of the second, as the first is to the third: For $aa : aarr :: a : arr$, or $4 : 16 :: 2 : 8$.

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b

Examp.

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INTRODUCTION. xxxiii

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b

Examp.

Examp. If 36 men perform 786 yards of work, how many yards will 100 men perform at the same time? Then $36 : 786 :: 100 : 2183\frac{1}{3}$ yards.

Examp. If 200 men perform 500 yards in 10 days, how many yards will 120 men perform in 180 days? Say 200 men is to 120, as 500 yards is to 300 yards, performed in the same time: and 10 days is to 180 days as 300 yards is to 2400 performed by 120 men in 180 days.

If the number to be found is greater or less than the given one of the same name; *place the least or greatest of the other two in the first place.* Thus if 46 men gained a 100 pounds, how many men will gain 550 pounds? As 550 is greater than 100, the number of men sought must be greater than 46. Say 100 pounds is to 550, as 46 men is to 253 the number sought. Again, if 2 yards of cloth cost 15 shillings, how much will cost 60 yards? As two yards is less than 60; 15 shillings must be less than the number sought: Say 2 yards is to 60, as 15 shillings is to 450 the number of shillings wanted.

Of EQUATIONS and their Reduction.

§. 38. The equality between two quantities consisting of one or more terms is called an *Equation*, as $a+x=b$, or $ab=cx$; the value of an unknown quantity is said to be found, when it is on one side of the equation, and all the known ones on the other.

The known quantities are expressed by the initial letters *a, b, c, d* of the alphabet, and the unknown or sought ones, by the final *u, x, y, z*. The several operations whereby the value of the unknown quantity is found, are called *Reductions*.

REDUCTION by ADDITION and SUBTRACTION.

§. 39. If a known quantity is on the same side with the unknown one; *take it away and place it on the other side with a contrary sign.* Thus if $x-4=6$, we get $x=6+4$ or 10; or if $x-a=b$, then $x=a+b$.

For to take away a quantity from one side, and place it

it on the other with a contrary Sign, is no more than to add or subtract, equals to or from equals; which does not destroy the equality. If the unknown quantity be negative, it must be transposed to the other side; thus if $6 - x = 4$, then $6 = 4 + x$, and $6 - 4$, or $2 = x$; or if $a - x = b$, then $a = b + x$, or $a - b = x$.

REDUCTION by MULTIPLICATION and DIVISION.

§. 40. If the unknown quantity be multiplied or divided by a known one, take it away, and divide or multiply all the known terms by it.

If $2x = 4a + 6b$, then $x = 2a + 3b$, by division; if $\frac{1}{3}x = a + c$; then $x = 3a + 3c$, by multiplication. For to take away a multiplier or divisor from one term, and divide or multiply all the others, is no more than to divide or multiply, equals by equals, which does not affect the equality.

If $\frac{1}{3}x - 4 = 10$, then $\frac{1}{3}x = 14$ by addition, $2x = 42$ by multiplication, and $x = 21$ by division. If $6 - \frac{1}{3}x = 3$, then $3 = \frac{1}{3}x$ by transposition, $12 = x$ by multiplication, and $x = 4$ by division; and if $\frac{1}{3}x = 46 - \frac{1}{2}x$, multiplying by 4×6 , we get $6x = 1104 - 4x$, or $6x + 4x = 1104$; and $x = 110.4$.

REDUCTION by the Roots or Powers.

§. 41. The square root of a quantity as $a + b$, is denoted by $\sqrt{a + b}$; thus if $\sqrt{a - x} = c$, then $a - x = cc$, by squaring both sides, or $a - cc = x$, if $\sqrt{4x + 5} = 5$, then the square gives $4x + 5 = 25$, or $4x = 20$, and $x = 5$.

If $xx + 2ax = bc$; by adding the square aa of half the coefficient of x to both sides, the first $xx + 2ax + aa$ will be a perfect square, whose root (§. 20) is $a + x = \sqrt{bc + aa}$.

Hence 1°. If the second term $2ax$ is negative, then $x - a = \sqrt{bc + aa}$, or $a - x = \sqrt{bc + aa}$, according as x is greater or less than a , which is always known by the nature of the question.

2°. If a is greater than x , the second side bc must be negative, and less than the square aa of half the coefficient of x ; for since $a - x = \sqrt{bc + aa}$ gives $a - \sqrt{bc + aa} = x$; and as the root of $bc + aa$ is greater than a , when bc is positive, the value of x would be negative: And when bc is negative, and greater than aa , the root of $bc + aa$ will be impossible; since the square root of a negative quantity cannot be extracted.

3°. If the second term $2ax$ is wanting, then $xx = bc$, gives $x = \sqrt{bc}$, or $-x = \sqrt{bc}$; for either of these roots squared, gives $xx = bc$.

4°. Every square has two roots, either both possible, or both impossible; for the equation $xx + 2ax = bc$, gives $x + a = \sqrt{bc + aa}$ or $x + a = -\sqrt{bc + aa}$, for the square of either gives the same equation; and when bc is negative and greater than aa , both roots are impossible: the negative roots are only useful in geometry.

When the highest power of the unknown quantity is the square, as in the equation above, it is called *Quadratic*. Hence all quadratic equations may be reduced to this form $xx + 2ax = bc$. For if the square xx has any coefficient, it may be freed from it by §. 40.; and if any term be repeated more than once, its coefficients may be added together.

This is as much algebra as a military gentleman requires to understand all my works: those who are desirous to know more of it, may consult other authors; such as Mac Laurin, Saunderson, and Tho. Simpson.

Errata to the Introduction.

Page	line	for	read
6	31	$a > c$	$a < c$
7	3	$+a = -b$	$a = b$
11	5	$-6ab, -24ax$	$+6ab, +24ax$
11	6	operations	operation
14	10	$-2ab$	$-a$
14	18	$+\frac{7ac}{2d}, -\frac{2aab}{cd}$	$-\frac{7ac}{2d}, +\frac{2aab}{cd}$
15	22	$a + b$	$a - b$

ELEMENTS

MATHEMATICKS.

BOOK I.

GEOMETRY.

SECT. I.

LINE S and ANGLES.

DEFINITIONS.

1.  GEOMETRY is that part of Mathematicks, which compares such quantities together as have extention.

2. Extension is distinguished into length, breadth, and thickness, which are called Dimensions.

A Term, or Bound, is the extreme of any thing.

3. A Line has only length, as AB, CD.

4. The terms or bounds of a line, are called Points, and have no dimensions.

5. A Surface has length and breadth only, as C.

The bounds of a surface are lines.

6. A Solid has length, breadth, and thickness, as D.

The bounds of a solid are surfaces.

7. A Right Line lies evenly between its terms, or is the shortest distance from one point to another, as AB.

And a Curveline, that which lies unevenly between its terms, as CD.

8. The opening A of two right lines BC and DC, which meet in a point C, is called an Angle, and is generally expressed by three letters BCD, with that placed at the angular point in the middle.

N. B. *This definition only regards the difference between the angles and not their measures; thus the angle ECD is said to be less than the angle ECB, because the opening ECD, is less than ECB.* Fig. 1.

9. If a right line CD meets another AB, and the angles CDA, CDB, on both sides are equal, that line CD is said to be perpendicular to the line AB.

10. And the equal angles, are called Right Angles.

11. But if a right line ED meets obliquely another AB; then the greatest angle ADE, is called an Obtuse Angle, and the least EDB, an Acute Angle.

There are therefore three different angles; the right, the obtuse greater, and the acute, less than the right one.

12. If two right lines AE, BD, meet another right line AC, so as to make the angles EAC, DBC on the same side equal, these lines, are said to be parallel.

Quantity, is whatever is capable of being estimated, numbered or measured; and magnitude a continued quantity, consisting in lines, angles, surfaces or solids.

A X I O M S.

1. Quantities, which are equal to one and the same quantity, are equal to each other.

2. If equals are added to, or taken from equals, the wholes or remainders will be equal.

3. If equals are added to, or taken from unequals, the wholes or remainders will be unequal, and that which was the greatest will remain the greatest.

4. Magnitudes which coincide with one another, or exactly fill the same space, are equal.

All rectilineal figures, whose sides and angles being respectively equal, are then equal.

5. The whole is equal to all its parts taken together.

5. The

6. The doubles, halves, or any like parts, of equals are equal.

7. All right angles are equal to one another.

P E T I T I O N S.

1. Grant, that a right line may be drawn from one given point to another.

2. That a given right line may be produced at pleasure.

3. That from a given point a right line may be drawn, either parallel to another, or so as to make a given angle, with any other right line.

N. B. We shall not refer to these demands in this work, because the reader will easily see, where they are used. And for brevity, we shall hereafter call a right line, a line only.

T H E O R E M I.

1. If a line ED meets another line AB , the angles EDA , EDB , at the same side of it, will together be equal to two right angles. Fig. 1.

If ED be perpendicular to AB , each of the angles EDA , EDB , will be a (*def. 9.*) right angle; and if the line ED be oblique to AB , let CD be perpendicular to AB : then the greatest angle ADE , exceeds the right angle ADC , by the angle CDE , or by as much as the least angle EDB wants of it; it is evident that these angles, are together equal to two right angles.

C O R.

2. Hence, if several lines meet another line AB in the same point D ; all the angles, on the same side of that line, are equal to two right angles; since the whole is equal to (*ax. 5.*) all its parts taken together.

T H E O R E M II.

3. If two lines AB , CD , cross each other, the opposite angles a , c , are equal. Fig. 2.

Because the line DE meets AB in E , the angles a , b , are equal to two right (*art. 1.*) angles; and since BE

B 2

meets

meets CD in E , the angles b , a , are also equal to two right angles; the angles c , b , (*ax. 1.*) are therefore equal to the angles b , a , and by taking away the angle b , which is common to both, the angle c will be (*ax. 2.*) equal to the opposite angle a .

C O R.

4. Hence it is manifest, that if ever so many lines cross each other in the same point, all the angles at that point will be equal to four right angles; since all those on one side of a line are equal to two right angles, (*art. 2.*) as well as all those on the other.

T H E O R E M III.

5. If two parallel lines AB , CD , are crossed by any other line EF ; the alternate angles AGF , DHE , are equal. Fig. 3.

The opposite angles a , c , are (*art. 3.*) equal, as well as the angles b , c , on the same side (*def. 12.*) by supposition; since both the angles a and b , are equal to the same angle c , they are therefore equal (*ax. 1.*) to each other; and it is proved in the same manner, that the alternate angles BGF , and CHE , are equal.

C O R I.

6. Hence, if two lines are intersected by another line, and the alternate angles are equal; these lines are parallel to each other; for since the angle a is equal (*art. 3.*) to its opposite one c , and to the alternate one b , by supposition, the angles c and b , at the same side being equal to the same angle a , are (*ax. 1.*) equal to each other; the lines AB , CD , are (*def. 12.*) therefore parallel.

C O R II.

7. It appears also, that if a line EF be perpendicular to a line CD , it will be so to all parallels to that line; for since b is a right angle by supposition, its equal c will be (*def. 12.*) a right angle likewise; consequently EF is perpendicular to both lines AB , and CD .

S E C T.

S E C T O R II.

P L A N E T R I A N G L E S.

D E F I N I T I O N S.

13. **A** Figure is a space contained between one or more terms.

14. A plane figure is that which lies evenly between its terms, or agrees with a right line applied on it according to any direction.

15. A plane figure, terminated by three lines, is called a Triangle.

16. When the three sides are equal, it is called an Equilateral Triangle.

17. When two sides are only equal, an Isosceles Triangle.

18. When one angle is a right one, a Right Angled Triangle.

19. Any side of a triangle may be called the Base; then the angle opposite to it, is called the Vertex.

20. The side opposite to a right angle, is called Hypothenuse.

There are therefore three particular sorts of triangles, the Equilateral, the Isosceles, and the Right-angled.

T H E O R E M I.

8. If any side AC of a triangle ABC , be produced, the external angle BCD , will be equal to the two internal opposite ones A and B . Fig. 4.

Let the line CE be parallel to the side AB , then the angles A and a , at the same side are (*def. 12*) equal; and because the line BC meets two parallel lines AB , CE , the alternate angles B and b are (*art. 5.*) equal. Therefore the internal angles A and B are equal to the two angles a and b ; that is, to the external angle BCD .

C O R. I.

9. Hence, the external angle BCD , is always greater than either of the two internal opposite ones, A or B , since it is equal to both.

C O R O L L.

10. Therefore, parallel lines can never meet, tho' produced ever so much; for while the lines AB , CB meet, the external angle BCD will always be greater than the internal opposite one A , which is contrary to the definition of parallel lines.

C O R. III.

11. It appears also, that if the two internal opposite angles A and B , are equal; the external angle BCD , is double, either of these angles A or B .

T H E O R E M II.

12. *The three angles of any triangle, are together equal to two right angles.*

The external angle BCD is equal (*art. 8.*) to the two internal opposite ones A and B ; to these equals, add the angle ACB ; then the angles BCD , BCA , will be equal to the three angles A , B , C , of the triangle, as well as to two (*art. 1.*) right angles; therefore the three angles of any triangle are together equal (*ax. 1.*) to two right angles.

C O R. I.

13. Fig. 5, 6. Hence, if two triangles ABC , DEF , have two angles A , B , in the one, equal to two angles D , E , in the other; the third C , in the one, will be equal to the third F in the other; as being each the difference between two right angles (*art. 12.*) and two equal ones.

N. B. It does not readily appear from the definition of parallel lines, that when two lines EH , ED , Fig. 3. are differently inclined to two parallel lines AB , CD , they make each the angles at the same side equal; but if the angles b , c , at the same side of the one, are equal; then the triangles EGB , $EH D$, having the angles b , c , equal, and the angle E common; the third EBG in the one will be equal to the third EDH in the (*Art. 13.*) other.

C O R.

C O R. II.

14. It appears also, that if one angle of a triangle be either equal to, or greater than a right angle, either of the other angles will be less than a right angle; since they are together equal to, or less than (*art. 12.*) a right angle.

T H E O R E M III.

15. *Two triangles, having one angle equal each to each, as well as the adjacent sides, are equal in all respects; that is, the third side in the one is equal to the third side in the other, and the angles opposite the equal sides are equal.*
Fig. 5, 6.

If the angle A be equal to the angle D , the side AB to the side DE , and AC to DF , imagine the triangle DEF placed on the triangle ABC , so as the side DE agrees with its equal AB ; then the angle D will agree with its equal A , and the side DF with its equal AC ; and since the points E and F , coincide with the points B and C ; the side EF , will also coincide with the side BC ; these triangles are therefore (*ax. 4.*) equal in all respects; that is, the third side EF is equal to the third side BC , the angle E to the angle B , and the angle F to the angle C , opposite to equal sides.

T H E O R E M IV.

16. *In an isosceles triangle ABC , the angles C, A , opposite to the equal sides BA, BC , are equal.* Fig. 7.

Let the line BD bisect the angle B ; then the triangles BDA, BDC , having the equal sides BA, BC , the side BD common to both, and the angles at B , included by these equal sides, equal by construction; are (*art. 15.*) equal in all respects; the angles C, A , opposite to the common side BD , are consequently equal.

C O R. I.

17. Fig. 8. Hence, the three angles of an equilateral triangle ABC are equal. For the sides AB, AC , being equal, the opposite angles C, B , are equal; and the

sides AB , BC , being equal, the angles C and A are equal: therefore the three angles A , B , C , of an equilateral triangle (*ax. 1.*) are equal.

C O R. II.

18. Fig. 7. It is evident that the line BD , which bisects the angle B , contained by two equal sides BA , BC , bisects likewise the opposite side AC at right angles. For the triangles BDA , BDC , being equal, the angles at D opposite to the equal sides BC , BA , are equal, they are therefore (*def. 9.*) right angles, and the sides AD , DC , opposite to the equal angles at B , are equal.

C O R. III.

19. Two isosceles triangles, which have one angle at the base or at the vertex equal, are equiangular. If it be the angles at the vertex; the remaining two or their halves in the one, (*art. 12.*) will be equal to the remaining two, or their (*art. 16.*) halves in the other: and if it be an angle at the base, the others at the base are also equal, and consequently, the third in the one (*art. 13.*) is equal to the third in the other.

C O R. IV.

20. Lastly, each of the angles at the base of an isosceles triangle, will always be less than a right angle, (*art. 12.*) and consequently, its external angle, greater than a right angle.

T H E O R E M V.

21. Two right angled triangles BDA , BDC , having the hypotenuses BA , BC , equal, and any of the other sides BD , are equal in all respects.

For the sides BA and BC being equal, the opposite angles C and A (*art. 16.*) are equal, and the angles at D being right angles by supposition, the angles at B (*art. 13.*) are equal. Consequently, the sides BA , BC , and BD , including these equal angles, being equal, these triangles are equal in (*art. 15.*) all respects.

C O R.

Hence, no more than one line BD can be drawn from the same point B , perpendicular to the same line AC .

THEOREM VI.

22. *In any unequal sided triangle, the greatest angle is opposite to the greatest side, and the greatest side opposite to the greatest angle.* Fig. 9.

In the greatest Side AC , take AD equal to a less side AB , and draw BD : then because AD is less than AC , the line BD falls within the triangle ABC ; and as AD is equal to AB , by construction the angles ADB and ABD (*art. 16.*) are equal; but the external angle ADB to the triangle BDC , is greater (*art. 9.*) than the internal opposite one C , and its equal ABD , less than the angle ABC ; the angle ABC opposite to the greatest side, is consequently greater than the angle C opposite to a less side AB .

The contrary is likewise true; for if the angle ABC be greater than the angle C ; the angle ABC must be greater than the angle ABD , otherwise the angle C would be equal to it; which is contrary to supposition; therefore the side BC falls without the triangle ABD , and the side AC opposite to the greatest angle ABC , is greater than the side AD or its equal AB opposite to a less angle C .

C O R.

23. Hence the sum of any two sides of a triangle ABD , is always greater, and the difference less than the third side; produce any side AD , so as DC be equal to DB , and join BC . Because DC is equal to DB , the angle DBC is equal to (*art. 16.*) DCB ; and as the angle ABC is greater than DBC , it is also greater than DCB ; and since the greatest side of a triangle is opposite to the (*art. 22*) greatest angle, AC , or the sides AD and BD , are together greater than the side AB . If we take DB from AB , and its equal from AC ; the

the difference between the sides AB and BD is less (*ax. 3.*) than the other side AD .

THEOREM VII.

24. *If two triangles have two sides equal each to each, that which has the greatest angle between these sides, will have the greatest base; and the greatest base will be opposite to the greatest angle.* Fig. 10, 11.

In the triangles DEF , ABC , let the side DE , be equal to the side AB , the side EF to the side BC , and the angle DEF greater than the angle ABC .

Imagine the side DE , not greater than EF , placed on its equal AB , and let BH represent EF , and AH the base DF , join CH . The sides BC , BH being equal by supposition, the angles BCH , BHC (*art. 16.*) are equal; but the angle ACH is greater than the angle BCH , and the angle AHC , less than its equal BHC ; the angle ACH is greater than the angle AHC ; therefore AH or its equal DF , is (*art. 22.*) greater than AC ; consequently, the greatest base DF is opposite to the greatest angle E .

The contrary is also evident; for if AH is greater than AC , the angle ACH , opposite to the greatest side AH is greater (*art. 22.*) than the angle AHC . Therefore the line BH , falls without the triangle ABC , and so the angle ABH , opposite to the greatest side AH , is greater than the angle ABC .

THEOREM VIII.

25. *Two equiangular triangles, which have one side equal and opposite to equal angles, are equal in all respects.* Fig. 12, 13.

If the side AC , be equal to the side DF , the angle A to the angle D ; the angle B to the angle E , and the angle C to the angle F ; imagine the triangle DEF , placed on the triangle ABC , so that the side DF agrees with its equal AC , then the angle D being equal to the angle A , the side DE will agree with the side

side AB , and since the angle F is equal to the angle C , the side FE will also agree with the side CB , and therefore the angle E will coincide with its equal B . Consequently the side DE will be equal to the side AB , and the side FE to the side CB .

THEOREM IX.

26. If two angles C and A of a triangle ABC are equal, the sides BA , BC , opposite to these equal angles, are equal. Fig. 7.

Let BD be perpendicular to AC ; then the right angled triangles BDA , BDC , having, besides the right angles at D , the angles A and C , equal by supposition, and the side BD common to both; they are equiangular and (*art. 25*) equal in all respects; the sides AB , BC , opposite to the right angles, are consequently equal.

COR.

27. Fig. 8. Hence, if the three angles of a triangle are equal, the sides are likewise equal. For the angles A and C , being equal, the opposite sides BC , BA , are equal; and the angles A and B being equal, the opposite sides BC , AC , are equal. Therefore, when the three angles of a triangle are equal, the three sides are also equal.

THEOREM X.

28. If the three sides of one triangle are equal to the three sides of another, respectively; these triangles are also equiangular. Fig. 12, 13.

If the side AB be equal to the side DE , the side AC to DF , and BC to EF , then the angle A will be equal to the angle D , the angle B to the angle E , and C to F . For if the angle B be either greater or less than the angle E ; the adjacent sides to these angles being equal by supposition; the base AC , opposite to the angle B , will also be (*art. 24*.) greater or less than the base DF , which is contrary to supposition. Since therefore the angle B can neither be greater nor less than the angle

gle E, it must be equal to it. These triangles are consequently (*art. 15.*) equal in all respects; that is, the angle A is equal to the angle D, and the angle C to the angle F.

THEOREM XI.

29. *If two triangles have two sides respectively equal, and one angle to one angle; and if the perpendiculars drawn to the other sides, fall both either within or without the triangles; they will be equal in all respects.* Fig. 14. 15.

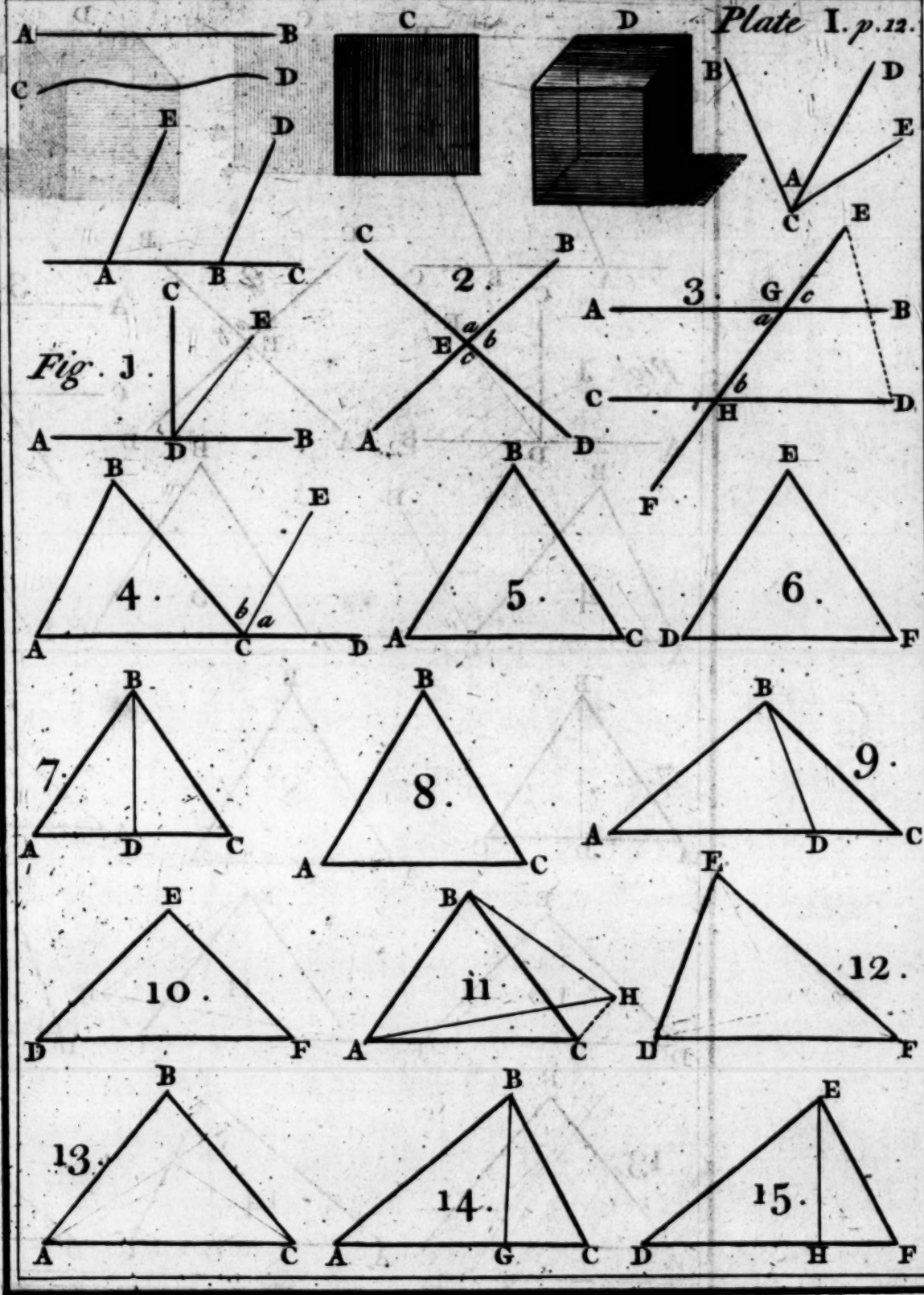
If the side AB be equal to the side DE, BC to EF and the angle A to the angle D; draw BG, EH, perpendicular to AC and DF; then the right angled triangles AGB, DHE, having, besides the right angles, the angles A, D, equal, as well as the sides AB, DE, by supposition, are equiangular and (*art. 25*) equal in all respects; that is, AG equal to DH, and BG to EH.

And the right angled triangles BGC, EHF, having the sides BG, EH, equal as well as the hypotenuses BC, EF; by supposition, are (*art. 21.*) equal in all respects; that is, GC is equal to HF, the angle C to the angle F, and therefore AC (*ax. 2.*) equal to DF, and the angle ABC (*art. 13.*) equal to the angle DEF.

LEMMA R K.

It appears from the articles 15, 25, 28 and 29; that if two triangles have one angle, and the adjacent sides to these angles equal, or two angles and one side; the three sides, or two sides and one angle, respectively equal; these triangles are equal in all respects; only it must be known, in the last case, that the perpendiculars drawn to the other sides, fall either both within or without the triangles, or else that case is not determined.

ing equal by supposition; the side AC, opposite to the angle B, will also be (ax. 24.) greater or less than the side DE, which is contrary to supposition. Since therefore the angle B can neither be greater nor less than the angle E, it must be equal to it. **SECT.**



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S E C T III.

PARALLELOGRAMS and TRIANGLES.

D E F I N I T I O N S.

1. **A** Plane figure of four sides is in general, called a Quadrilateral.
2. A parallelogram is a quadrilateral, whose opposite sides are parallel.
3. A rectangle is a parallelogram, whose four angles are equal.
4. A square is a parallelogram, whose four sides are equal, as well as the four angles.
5. All other quadrilaterals, are called Trapeziums.
6. A diagonal, is the line which joins the opposite angular points of a quadrilateral.
7. If two lines parallel to the sides of a parallelogram, intersect each other in the diagonal, they form two parallelograms within the first on each side of the diagonal, which are called Complements.
8. The altitude of a parallelogram, is the perpendicular distance between two opposite sides.
9. Either of the sides to which the altitude is perpendicular, is called the Base.
10. The altitude of a triangle, is the perpendicular drawn from the vertex to the base; and the parts of the base terminated by the perpendicular, are called Segments. Therefore two triangles, whose bases are in the same right line, and which have the same vertex, have the same altitude, as being expressed by the same perpendicular.

T H E O R E M I.

30. *The opposite sides of a parallelogram are equal, as also the opposite angles, and the diagonal bisects the parallelogram.* Plate II. Fig. 1.

The

The opposite sides AD , BC , as well as AB , DC , are parallel by definition; and hence the alternate angles ADB , DBC , are (*art. 5.*) equal, as well as the alternate ones ABD , BDC . Therefore, the triangles ADB , DBC , having the common side BD , and the adjacent angles equal, are (*art. 25.*) equal in all respects; that is, the side AD is equal to the side BC , opposite to equal angles, and the side AB to DC . The angles A , C , opposite to the common side BD are equal, as well as the angles ABC , ADC , as being the sum of equal ones.

C O R. I.

31. Hence it is manifest, that the lines AB , DC , which join the extremities of two equal and parallel lines AD , and BC , on the same side, are likewise equal and parallel.

C O R. II.

32. Since the three angles of a triangle are equal to (*art. 12.*) two right angles, and the four angles of a quadrilateral, equal to the angles of two triangles; the four angles of a quadrilateral are (*ax. 1.*) equal to four right angles.

C O R. III.

33. As the four angles of a quadrilateral, are equal to four right angles, and the angles of a square or rectangle, equal by definition; the angles of a square or rectangle, are right ones.

C O R. IV.

34. Since the opposite sides of a rectangle are equal and perpendicular to the other opposite ones which are parallel: it follows, that parallelograms, which are between the same parallels, have the same altitude.

C O R. V.

35. It is manifest, that any triangle ADB , is always half the parallelogram AC , of the same base AD , and the same altitude.

C O R.

C O R. VI.

36. Consequently, triangles, which are between the same parallels, have equal altitudes; since they are the same as those of parallelograms, which are between the same parallels.

T H E O R E M II.

37. *The diagonals AC, DB, of a parallelogram, bisect each other. Fig. 2.*

Let E be their intersection; then the sides AD, BC, being equal, and parallel, (*art. 30*) and the alternate angles EAD, ECB, are also equal as well as the opposite ones at E; the triangles AED, BEC, are (*art. 25*) equal in all respects; consequently the sides AE, EC, and BE, ED, opposite to equal angles, are equal.

T H E O R E M III.

38. *The complements AI, IC, of a parallelogram AC, are equal. Fig. 3.*

Because the diagonal BD, divides the parallelograms AC, FG, EH, each into two (*art. 30*) equal triangles; by taking the triangles BFI, IED, from the triangle BAD; and their equals, BGI, IHD, from the triangle BCD, equal to BAD; we shall have the complement AI equal (*ax. 2.*) to the complement IC.

T H E O R E M IV.

39. *Parallelograms, which have the same or equal bases, and the same altitude, or are between the same parallels, are equal. Fig. 4.*

First, Let the parallelograms DB, AF, have the same base AD, parallel to BF; then BC and EF being both equal to AD (*art. 30*) are equal to each other, as well as BE and CF (*ax. 2.*); and since AB, DC are parallel and equal (*art. 30*) the angles ABE, DCF on the same side are equal (*def. 12*). Hence the triangles ABE, DCF, having the angles at B and C equal as well as the adjacent sides, are (*art. 15*) equal in all respects:

Whence,

Whence, if the triangle ABE be taken from the trapezium ABFD we shall have the parallelogram AF, and if the triangle DCF be taken from the same trapezium we shall have the parallelogram DB. Therefore the parallelograms AF, DB are equal by *ax. 2.*

Secondly, Fig. 5. If the base AD be equal to the base EH, and BG parallel to AH, draw AF and DG, then as the side EH is equal to its opposite FG, and to AD by Supposition, AD is equal and parallel to FG, therefore AG is a parallelogram (*art. 31*) and equal to both parallelograms DB, and EG, by the first case; consequently, the parallelogram AC is (*ax. 1*) equal to the parallelogram EG.

C O R. I.

40. Since triangles are the halves (*art. 35*) of parallelograms of the same base and altitude, and as all parallelograms of equal bases and altitudes are equal; it is manifest, that all triangles which have equal bases and altitudes, or are between the same parallels, are likewise equal by *ax. 6.*

C O R. II.

41. Hence fig. 6. if AG be a rectangle, it is manifest that any parallelogram FC, is always equal to a rectangle of the same base and altitude; and since a triangle is half the parallelogram of the same base and altitude; a triangle is therefore also half the rectangle of the same base and altitude.

C O R. or T H E O R E M V.

42. If from the point D a line DE be drawn perpendicular to AC, the right angled triangles ABC, and DEC, having the angle at C common to both, are (*art. 13*) equiangular; and as the parallelogram FC is equal to the rectangle AG (*art. 41*) as well as to the rectangle of the same base DF and the same altitude DE; the rectangle AG is equal to the rectangle made of DF and DE (*ax. 1*); that

that is the rectangle made by the hypotenuse DC or FA of the one, and the side AB of the other, is equal to the rectangle made by the hypotenuse FD or AC of the other, and the side DE in the first opposite to the same angle C.

THEOREM VI.

43. *If a line AD be divided into any two parts, AE, ED, the square BD of the whole line, will be equal to the squares and two rectangles of the parts. Fig. 7.*

In AB, the side of the square, take AG, equal to the part AE, draw GH parallel to AD, and EF parallel to AB; then as AG is equal to AE by construction, and all the angles of the figure right angles by supposition, EG is the square of AE; and since AD is equal to AB, and AE to AG, GB or HC, is (ax. 2.) equal to ED or FC; therefore FH is the square of ED; GF and EH, rectangles made by the parts AE and ED. The square BD of the whole line, is consequently equal to the squares EG, HF, and to the two rectangles GF, EH, of the parts AE, ED.

THEOREM VII.

44. *The difference between any two squares ABCD, ILCK, is equal to the rectangle made by the sum and difference of their sides. Fig. 8, 9.*

In AD produced, take DG equal to the side IK or ED of the square KL; produce IK, and draw GF perpendicular to AG: Then DG being equal to DE, by construction, AG will be equal to the sum of the sides; and as HF is parallel to AG, AH will be equal to the difference. Whence the rectangle AHFG, is made by the sum and difference of the sides. But DG being equal to DE, by construction, the rectangle DF is equal to the rectangle DI or its complement IB. Therefore, the difference HD and HL, between the squares ABCD, and ILCK, is equal to the rectangle AGFH, made by the sum and difference of the sides.

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C O R.

45. Hence it is manifest, that if a line EG be equally divided in D , and inequally in A , the rectangle $AHFG$, made by the unequal parts AE , AG , will be equal to the difference between the squares DB , KL , made by half that line and the part AD . For since ED or LC is equal to DG by supposition, as is AE to AH , by construction; the rectangle DF is equal to the rectangle KE or HI . Consequently, the rectangle $AHFG$ is equal to the difference DH and HL , between the squares DB and KL .

T H E O R E M VIII.

46. *The Rectangle AC , made by any two lines AD and AB , is equal to the sum of all the rectangles, made by one of these lines AB , and the parts of the other AD divided any how.* Fig. 10.

Let the line AD be divided into any parts AE , EG , GD ; and thro' these divisions, draw EF , GH , parallel to AB ; then because BC being parallel to AD , the lines EF , GH are each equal to AB ; therefore the rectangles AF , EH , GC , are made by the line AB and the parts AE , EG , GD , of the line AD , and equal (*ax. 5.*) to the rectangle AC .

T H E O R E M IX.

47. *In any right angled triangle ACB , the square $ABED$, of the hypotenuse AB , is equal to the squares CH , CF , of the two other sides AC , CB .* Fig. 11.

Thro' the right angle C , draw GL perpendicular to AB ; then the right angled triangles, ACB , ALC , having the angle A common to both are (*art. 13.*) equiangular, and so the angle ACL is equal to the angle ABC ; therefore the rectangle LD , made by the hypotenuse AB or AD , and the side AL , is equal to (*art. 42.*) the rectangle or square CH , made by the hypotenuse AC and the side AC , opposite to the angle B equal to the angle ACL .

The

The triangle CBL being equiangular to the triangle ACB ; it may be proved in the same manner, that the rectangle LE , is equal to the square CF of CB . But the rectangles LD , LE , are equal to the square AB ; therefore the square of the hypotenuse is equal to (ax. 1.) the squares of the two other sides.

C O R. I.

48. Hence, in any right angled triangle, the difference between the squares of the hypotenuse and of any side, is equal to the square of the other side.

C O R. II.

49. It is manifest, that if a line AB , be divided into any two parts AL , LB ; the square AE of that line, is equal to the rectangles LD , LC , made of that line, and the parts into which it has been divided.

T H E O R E M X.

50. In any triangle ABC , where a perpendicular BD is drawn to the base AC ; the rectangle made by the sum and difference of the sides AB , BC , is equal to the rectangle made by the sum and difference of the segments AD , CD of the base. Fig. 12, 13.

In the right angled triangles BDA , BDC , the difference between the squares of AB and AD is (art. 48.) equal to the square of BD , and this square is also equal to the difference between the squares of BC and DC ; therefore the difference between the squares of AB and AD (ax. 2.) is equal to the difference between the squares of BC and DC ; to these equals add the square of AD , and subtract the square of BC ; then the difference between the squares of AB , and BC , will be equal to the difference between the squares of AD and DC . But the difference between two squares is equal to (art. 44.) the rectangle made by the sum and difference of their sides; consequently, the rectangle, made by the sum and difference of the sides AB and BC , is equal to the

rectangle, made by the sum and difference of the segments AD and DC .

T H E O R E M XI.

51. *Any trapezium, which has two sides AD , BC parallel, is equal to a rectangle of the same altitude, and whose base is half the sum of the parallel sides.* Fig. 14.

If EF be parallel to AB , and bisects DC in G , meeting BC produced in F ; the triangles GED , GFC , having the alternate angles D , C , (*art. 5.*) equal, as well as the opposite ones at G , and the side DG equal to the side GC , by construction; are (*art. 25*) equal in all respects; therefore the trapezium $ABCD$ is (*ax. 2.*) equal to the parallelogram $ABFE$. But AE is half the sum of the parallel sides AD , BC , and the parallelogram AE is equal to a rectangle of the (*art. 40.*) same base and altitude; therefore the trapezium is equal to a rectangle of the same altitude, and whose base is half the sum of the parallel sides.

S E C T. IV.

P R O P O R T I O N S.

This section is divided into two parts, the first contains the definitions and general Theorems, which comprehend chiefly what is said in the first book of *Euclid*: The second, the application of proportions to lines and surfaces, such as in his sixth book.

P A R T THE FIRST.

D E F I N I T I O N S.

1. **A** Less magnitude is said to be a part of a greater magnitude, when the less measures the greater exactly.

2. A magnitude is said to be parts of a magnitude, when it is measured by the same part as that magnitude.

3. Equal

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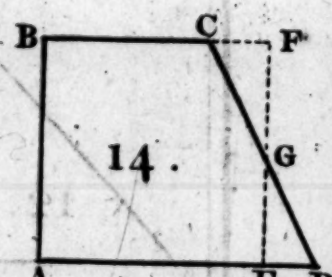
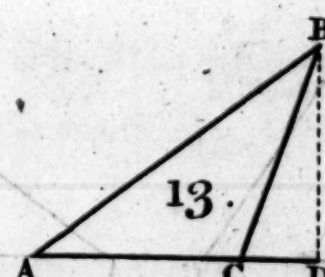
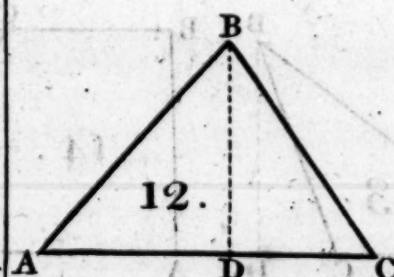
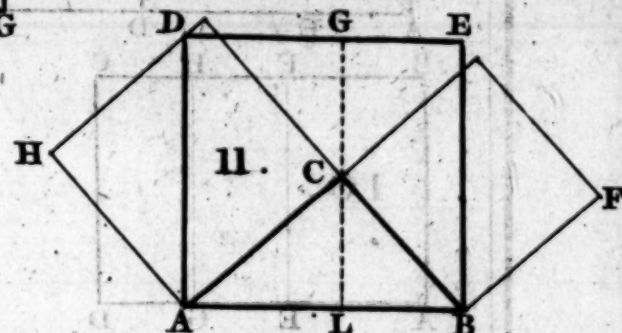
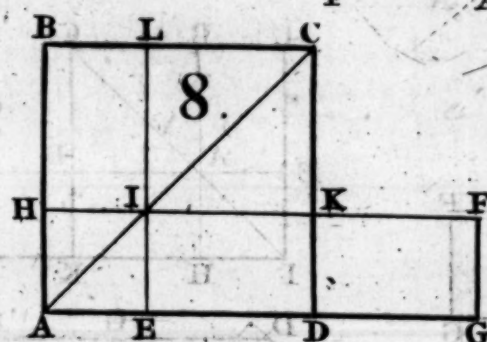
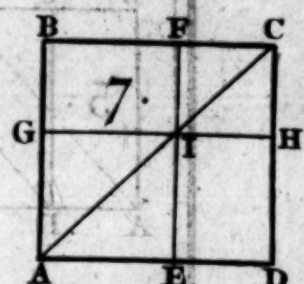
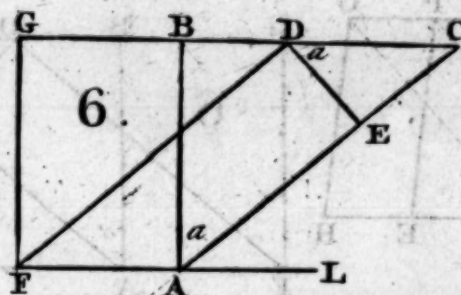
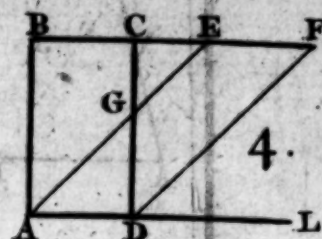
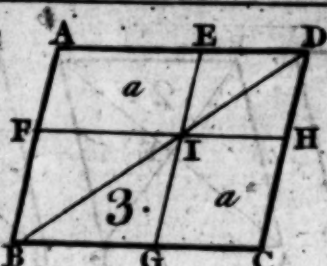
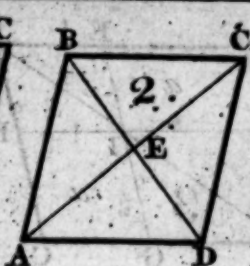
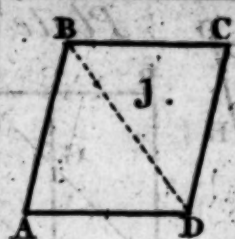
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3. Equal parts of two or more magnitudes, are the same number of equal parts, taken of them all.

Thus nA , nB , are said to be equal parts of the magnitude A and B, when n expresses any common fraction. Therefore, when the numerator is less, equal to or greater than the denominator; the equal parts nA , nB , are less, equal to or greater than the magnitudes A and B.

Thus the equal parts of 20 and 30, are 6 and 9, or 16 and 24: because 6 and 9 are the three tenths of 20 and 30, and 16, 24, the four fifths.

4. Ratio, is the relation between two magnitudes of the same kind, in respect to quantity.

In the comparing two magnitudes A and B, the first A is called Antecedent, and the second B Consequent.

In commensurable magnitudes the ratio is found by dividing the antecedent by the consequent. As for example, the ratio of 20A, and 5A, is 4; and in general the ratio of nA and mA is n divided by m .

5. Four magnitudes A, B, C, D, are said to be in the same ratio or proportion, the first A to the second B, and the third C to the fourth D; when any equal parts whatsoever nA , nC of the antecedents compared with their consequents B, D, respectively, are either both together, less, equal to, or greater than their respective consequents.

6. Magnitudes which have the same ratio are called Proportionals, and are marked thus, $A : B :: C : D$, which signifies that A is to B as C is to D.

7. Magnitudes are said to be in a continued proportion, when the consequent of any ratio is always equal to the antecedent of the succeeding ratio: As here $A : B :: B : C :: C : D$: the continued proportion is also marked thus $:: A : B : C : D$.

8. If magnitudes are in a continued proportion as $:: A : B : C : D$, the ratio of the first A to the third C is said to be Duplicate, and the ratio of the first A to the fourth D Triplicate, to that of the first A and second B.

On the contrary, the ratio of the first A to the second B, is said to be subduplicate to the ratio of the first A and third C, and subtriplicate to that of the first A and fourth D.

9. If there be any number of magnitudes of the same kind A, B, C, D, &c. the ratio of the first A to the last is said to be compounded of the first A to the second B, of the second B to the third C, of the third C to the fourth D, and so on to the last.

10. Equal magnitudes, are in the ratio of equality, unequal magnitudes in the ratio of inequality; the greater to the less in the ratio of the greater inequality, and the less to the greater in the ratio of the least inequality.

Thus if $20A$ be compared to $10A$, then the ratio of $20A$ to $10A$, is that of the greatest inequality, and the ratio of $10A$ to $20A$, that of the least inequality.

11. Inverse Ratio, is when the consequent is taken as antecedent, and the antecedent as consequent; thus if $A : B :: C : D$, then $B : A :: D : C$, is inversely.

12. Alternate Ratio, is when the antecedent of one ratio is compared to the antecedent of another, and the consequent to the consequent; as if $A : B :: C : D$, then $A : C :: B : D$, is alternately. But this comparison can only be made when the four magnitudes are of the same kind.

NOTES of ABBREVIATION.

$A=B$, signifies, that A is equal to B.

$A+B$, the sum of A and B.

$A-B$, their difference.

$A \times B$ or $A B$, a rectangle whose sides are denoted by A and B.

A^2 or $A A$, the square whose side is denoted by A.

N. B. *Though letters are used to express magnitudes, the reader is not to imagine, that we intend to demonstrate the doctrine of proportion by algebraic computations; As it is necessary to fix upon some symbol or other to represent magnitudes, in order to assist the mind, in distinguishing one from another*

ther, letters appear to be more easy than lines, as Euclid expresses them. But the reader may, if he pleases, mark lines over each letter.

S C H O L L I U M.

It is proper to observe, that there are three signs in the 5th definition of proportion, by which proportionality is discovered, because the comparison of magnitudes may be distinguished into three cases; for they may be always commensurable or incommensurable, or may be sometimes commensurable, and at others incommensurable, as in plain figures.

When magnitudes are commensurable, if any equal parts of the antecedents are equal to their respective consequents, that sign only is sufficient to prove the proportionality: Since, if any other equal parts of the antecedents are taken, they must necessarily be both together greater or less than the consequents.

And when magnitudes are incommensurable, as the radius to the circumference, it is sufficient to prove, that any equal parts of the antecedents, when compared with the respective consequents, are both together always greater or less than the consequents. For were it possible, that any parts whatsoever of one antecedent could be equal to its consequent, the same parts of the other antecedent would likewise be equal to its consequent; since it is manifest, that they can neither be greater nor less, unless the others are also greater or less.

But when magnitudes may either be commensurable or incommensurable, it is plain, that the three signs must be proved.

Hence it is evident, that our definition comprehends all possible cases that can happen in the comparison of all kinds of magnitudes whatsoever.

P E T I T I O N.

Grant that any magnitude may be, or conceived to be divided into any number of equal parts whatsoever. This is proved hereafter.

A X I O M S.

1. If equal parts of magnitudes are equal, the magnitudes themselves are equal; and the contrary.
2. If equal parts of magnitudes are unequal, the magnitudes are unequal; and that whose parts are greater will be greater; and the contrary.
3. Unequal magnitudes, compared with the same magnitude, have unequal ratios.
4. The halves are in the same ratio as the wholes.

T H E O R E M I.

52. *Two magnitudes A, B, which have the same ratio to a third C, are equal. If $A : C :: B : C$, I say, A is equal to B.*

If possible let A be unequal to B; then because unequal magnitudes cannot have the same ratio, when compared with the same magnitude, the ratio of A to C is not equal to the ratio of B to C, which is contrary to supposition; since therefore A can neither be greater nor less than B, they must consequently be equal.

T H E O R E M II.

53. *Two ratios which are equal to a third, are equal to each other. If $A : B :: E : F$, and $C : D :: E : F$; I say $A : B :: C : D$.*

By taking any equal parts nA , nE , nC , of the antecedents; then if $nA = B$, nE will be $= F$, and $nC = D$, by defin. 5; if nA is greater than B, nE is greater than F, and nC is also greater than D; and if nA is less than B, nE is less than F, and nC less than D. Therefore since the equal parts nA , nC , of the antecedents, in $A : B :: C : D$, when compared with the consequents, are both together, less, equal to, or greater than their respective consequents, these magnitudes are proportional by defin. 5.

T H E-

THEOREM III.

54. *Two magnitudes A, B, are in the same ratio as any of their equal parts. I say $rA : rB :: A : B$.*

By taking any equal parts nrA , nA , of the antecedents; then if $nrA = rB$, it is evident (ax. 1.) that $nA = B$; and if nrA , is either greater or less than rB : nA will also (ax. 2.) be greater or less than B. Consequently these magnitudes are proportional, by defin. 5.

C O R.

55. Hence, if $A : B :: C : D$; then because $nA : nB :: A : B$; and $mC : mD :: C : D$, we have $nA : nB :: mC : mD$, by (art. 53.) equality.

THEOREM IV.

56. *When four magnitudes, A, B, C, D, are proportional; I say, if the antecedents A, C, are equal, the consequents B, D, are equal; or if the first antecedent A be either greater or less than the second, its consequent B will also be greater or less than the consequent D.*

Because $A : B :: C : D$; if $A = C$, then will (art. 52.) B also be $= D$; and if A be greater than C, it is evident that B can neither be equal nor less than D; since if $B = D$, then A must be (art. 52.) equal to C; but A is supposed greater than C, therefore B cannot be equal to D, nor less; it must consequently be greater: By the same argument, if A is less than C, B is also less than D.

THEOREM V.

57. *If four magnitudes, A, B, C, D, are proportional, they will alternately be so. If $A : B :: C : D$; I say $A : C :: B : D$.*

By taking any equal parts nA , nB , of the antecedents; then because (art. 55) $nA : nB :: C : D$; and it has been (art. 56.) proved, that when the antecedents nA , C, are equal, the consequents nB , D, are equal; or if the first antecedent nA be either greater or less than the second C, its consequent nB will be also greater or less than

than the consequent D. Since therefore the four quantities A, C, B, D, are such, that any equal parts nA , nB , of the first and third, when compared with the second and fourth, are less, equal to, or greater than the second and fourth respectively, they are proportional by definition 5.

C O R.

58. Hence if $A : B :: C : D$, then alternately $A : C :: B : D$, and $nA : nC :: A : C$; or $nA : nC :: B : D$, by (*art. 53.*) equality; or alternately $nA : B :: nC : D$. By the same argument, $A : nB :: C : nD$, and by equality $nA : mB :: nC : mD$.

T H E O R E M VI.

59. If four magnitudes, A, B, C, D, are proportional, they will inversely be so. If $A : B :: C : D$. I say $B : A :: D : C$.

Because (*art. 58.*) $nA : mB :: nC : mD$; if we consider nA , nC , as any equal parts of the antecedents, when $nA = mB$, then (*def. 5.*) $nC = mD$; and when nA is greater or less than mB , nC is also greater or less than mD ; and therefore when mB is greater, equal to, or less than nA , mD is likewise greater, equal to, or less than nC .

Now if we consider mB , mD , as any equal parts of the antecedents in $B : A :: D : C$, or in $B : nA :: D : nC$; then, because it has been proved, that when mB is greater, equal to, or less than nA , mD was likewise greater, equal to, or less than nC ; it is evident that (*def. 5.*) these magnitudes are proportional. Consequently, if four magnitudes are proportional, they will inversely be so.

T H E O R E M VII.

60. If four magnitudes A, B, C, D, are proportional, the antecedent A will be to the antecedent C, as the sum of the first antecedent and its consequent is to the sum of the second antecedent and its consequent: If $A : B :: C : D$, I say $A : C :: A + B : C + D$.

For

For if any equal parts nA , $nA + nB$, of the antecedents are taken, because $nA : nB :: C : D$, if $nA = C$, then $nB = D$, and so $nA + nB = C + D$; if nA is either greater or less than C , nB will likewise be greater or less than D , and (*art. 58.*) therefore $nA + nB$, will likewise be greater or less than $C + D$. But if four magnitudes are such, that any equal parts of the first and third, when compared with the second and fourth, are greater, equal to, or less than the third and fourth respectively, they are proportional (*def. 5.*) consequently $A : C :: A + B : C + D$.

THEOREM VIII.

61. *If any number of magnitudes, A, B, C, D, E, F, are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents. If $A : B :: C : D :: E : F$. I say, $A : B :: A + C + E : B + D + F$.*

Because $C : D :: E : F$, we have (*art. 57.*) alternately $C : E :: D : F$, and (*art. 60.*) $C : D :: C + E : D + F$; but $A : B :: C : D$, by supposition: Therefore $A : B :: C + E : D + F$, by (*art. 53.*) equality; and by addition (*art. 60.*) $A : B :: A + C + E : B + D + F$.

THEOREM IX.

62. *If four Magnitudes, A, B, C, D, are proportional, they will be so by subtraction. If $A : B :: C : D$, I say $A : B - A :: C : D - C$.*

For if this proportion $A : B - A :: C : D - C$ be true, it will be so by (*art. 60.*) addition: But $A : B - A + A :: C : D - C + C$, or $A : B :: C : D$. Consequently, if four magnitudes are proportional, they will be so by subtraction.

COR.

63. Since $A : C :: B - A : D - C$ alternately, and (*art. 60.*) $A : C :: A + B : C + D$, we have $A + B : C + D :: B - A : D - C$, by (*art. 53.*) equality, or alternately $A + B : B - A :: C + D : D - C$.

It is to be observed, that $B - A$, $D - C$, are the differences between the antecedents and their respective con-

consequents; and therefore, when the antecedents are greater than their consequents, these differences are $A-B$, $C-D$; for since magnitudes which are proportional are inversely so, the demonstration (*art.* 62.) is the same in both cases.

T H E O R E M X.

64. *The sum of proportionals which have the same ratio are proportional. If $A:B::C:D$, and $E:F::G:H$, I say, $A+E:B+F::C+G:D+H$.*

By taking any equal parts nA , nC , nE , nG , of the antecedents, it is evident, that when $nA=B$, and $nE=F$, then $nC=D$, and $nG=H$, by supposition; there will also be $nA+nE=B+F$, and $nC+nG=D+H$; and when these parts are greater or less than their respective consequents, the sums will also be greater or less than the sums of their consequents: therefore, the sums of proportionals are proportional. The same thing holds true in respect to their differences.

P A R T II.

D E F I N I T I O N S.

1. Right-lined figures are said to be similar, when all the angles in the one are equal to all the angles in the other, and the sides between the equal angles are proportional, respectively.

2. The sides between the equal angles are said to be Homologous.

T H E O R E M XI.

65. *Parallelograms AE , BF , or triangles, which have the same or equal altitudes, are to each other, in the same ratio as their bases. I say $AB:BC::ADEB:BEFC$. Plate III. Fig. 1.*

Take BK , equal to any parts n of AB , draw KL parallel to CF ; then because parallelograms which have equal bases and equal altitudes are (*art.* 39.) equal, the
paral-

parallelogram BL , will be the same parts n of AE , as the base BK is of the base AB . But it is evident, that if BK is equal to BC , the parallelogram BL is equal to the parallelogram BF ; or if BK is less than BC , BL is also less than BF ; or if BK is greater than BC , BL is likewise greater than BF . Since then, the equal parts BK , BL , of the antecedents AB , $ADEB$, when compared with the consequents BC , $BEFC$, are both less, equal to or greater than their respective consequents; parallelograms which have the same or equal altitudes, are in the same ratio as their bases, by defin. 5.

And because the triangles AEB , BEC , are each half the parallelograms BD , BF , and the halves are in the same (*ax. 4.*) ratio as the wholes: triangles which have the same or equal altitudes, are in the same ratio as their bases.

C O R.

66. It is also manifest, that triangles BEC , BIC , or parallelograms, which have the same or equal bases BC , are to each other in the same ratio, as their altitudes BE , BI .

T H E O R E M XII.

67. *If in a triangle ABC , there be drawn a line DE , parallel to one of the sides BC , it will divide the others proportionally. I say, $AD:DB::AE:EC$, or $AD:AB::AE:AC$. Fig 2.*

Draw the lines DC , EB ; then because the triangles ADE , DEB , having the same vertex E , or the same altitude, are as (*art. 65.*) their bases; that is, $ADE:DEB::AD:DB$: and the triangles ADE , EDC , having the same vertex D , or altitude, are also as their bases; that is, $ADE:EDC::AE:EC$. But the triangles DEB , DEC , having the same base DE , and being between the same parallels DE , BC , (*art. 39.*) are equal: consequently, $AD:DB::AE:EC$, by equality of (*art. 53.*) ratios. The triangles AED , AEB , having

ing the same vertex E, or altitude, are as their bases, viz. $AED : AEB :: AD : AB$; and the triangles ADE, ADC, having the same vertex D, or altitude, are likewise as their bases, viz. $ADE : ADC :: AE : AC$; and since the triangles AEB, ADC, are each equal to two equal ones, they themselves are (*ax. 1.*) equal, consequently $AD : AB :: AE : AC$.

C O R.

68. It is manifest, that if the sides of a triangle are divided proportionally, the line joining the points of division, will be parallel to the other side. For since (*art. 65.*) $AD : DB :: ADE : DEB$: and $AE : EC :: ADE : EDC$: but $AD : DB :: AE : EC$, by supposition, therefore $ADE : DEB :: ADE : EDC$, by equality of ratios: whence the triangle DEB is (*art 53.*) equal to the triangle EDC; and as these equal triangles have the same base DE, they must (*art. 36.*) likewise have the same altitude: consequently DE is (*art. 36.*) parallel to the side BC.

T H E O R E M XIII.

69. *Equiangular triangles ABC, GHL, are similar; I say, $AB : GH :: AC : GL :: BC : HL$.*

In AB take AD equal to GH, and draw DE, parallel to BC: then because the angles A and G are equal as well as the angles B and H, by supposition, and the angle B is equal to the angle D on the same side, by the definition of parallel lines, the angle D is equal to the angle H. The triangles ADE, GHL, having the sides AD, GH, equal, as well as the adjacent angles by construction, are then equal (*art. 25.*) in all respects. Whence, $AB : AD$ or $GH :: AC : AE$ or GL .

If BD had been taken equal to GH, instead of AD, and the line DE drawn parallel to AC; it might have been proved in the same manner as before, that $AB : GH :: BC : HL$. Consequently equiangular triangles are similar.

T H E.

THEOREM XIV.

70. *If two triangles ABC, GHL, have one angle A, G, equal each to each, and the adjacent sides proportional, they are equiangular and similar.*

In AB take AD equal to GH, and in AC, take AE equal to GL, join DE; then the triangles ADE, GHL, having the angles A and G, equal by condition, as well as the adjacent sides by construction, they are equal in all (*art. 15.*) respects. And since $AD : AB :: AE : AC$, by construction; the triangle ADE, or its equal GHL, is equiangular and similar to (*art. 68.*) the triangle ABC.

THEOREM XV.

71. *Two triangles ABC, GHL, having their sides respectively proportional are similar.*

Suppose $GH : AB :: GL : AC :: HL : BC$; In AB, take AD equal to GH, and in AC, AE, equal to GL, join DE: then because $GH : AB :: GL : AC$, by supposition, and $AD : AB :: AE : AC$, by construction; the line DE is (*art. 68.*) parallel to BC. But (*art. 69.*) $AD : AB :: DE : BC$, and $GH : AB :: HL : BC$, by supposition, and since GH is equal to AD by construction, we have $HL : BC :: DE : BC$, by equality of ratios; whence HL is equal (*art. 53.*) to DE: the triangles GHL, ADE, having all their sides equal, are (*art. 28.*) equal in all respects, and consequently GHL is similar to ABC.

THEOREM XVI.

72. *If two triangles GHL, ABC, have one angle G, A, equal, and the sides adjacent to one of the other angles proportional, and the remaining angle of each, known to be either acute, right, or obtuse, they will be similar.*

Let $GH : HL :: AB : BC$: If in AB; the part AD, be taken equal to GH, and DE be drawn parallel

parallel to BC , we have (*art.* 69.) $AB : BC :: AD : DE$ as GH to HL , by supposition; and as AD has been taken equal to GH , we have DE also equal to HL ; besides the angles L and C are supposed to be both acute, right or obtuse. The triangle ADE or its equal $GH L$, is similar (*art.* 67.) to the triangle ABC .

T H E O R E M XVII.

73. *If four lines are proportional, the rectangle made by the first, and the last will be equal to the rectangle made by the means. Fig. 3.*

Take the side AB equal to the first line; and the perpendicular BC , to AB , equal to the second; compleat the rectangle BD ; draw the diagonal AC ; in AB take AE equal to the third, draw EG parallel to BC , and thro' its intersection I with the diagonal HF parallel to AB . Then because (*art.* 67.) $AB : BC :: AE : EI$; the fourth EI will (*art.* 52.) be equal to the fourth line. Now because the complements BI , ID , of a parallelogram, (*art.* 38.) are equal, by adding $E H$ to both; the rectangle BH made by the first and last, will be equal to the rectangle ED of the means.

C O R.

74. Hence, if three lines are in a continued proportion, the square of the mean will be equal to the rectangle made by the extremes. For if in the last construction, AE be taken equal to BC , ED will be the square of the mean, and still equal to the rectangle BH of the extremes.

C O R.

75. It is manifest, if four lines are such, that the rectangle BH of the first and last is equal to the rectangle ED of the means; these lines are proportional; since $AB : BC :: AE : EI$; and that if three lines are such, as the square of the mean is equal

to the rectangle made by the extremes; these lines are in a continued proportion.

C O R.

76. Hence, all that has been proved of magnitudes in general, in the first part of this section, follows in regard to lines,

Directly $A : B :: C : D.$

Inversely $B : A :: D : C.$

Alternately $A : C :: B : D.$

By multiplication $\left\{ \begin{array}{l} n A : n B :: C : D. \\ n A : B :: n C : D. \end{array} \right.$

By addition $A : A + B :: C : C + D.$

By subtraction $A : A - B :: C : C - D.$

By equality of ratio's $A + B : A - B :: C + D : C - D.$

For the four first changes are evident from that the rectangle of the means is (*art. 73*) equal to the rectangle of the extremes; the fifth and sixth from *art. 54*; and the last from *art. 53*.

T H E O R E M X V I I I.

77. If a line BD bisects any angle B made by the sides of a triangle, produced if necessary, it will divide the opposite side AC into parts proportionally to the other sides; I say, $AD : DC :: AB : BC.$ Fig. 4, 5.

In the side AB produced if necessary, take BE equal to the side BC, join EC; then the angles BEC, BCE, being equal, (*art. 16*) and each half the external angle CBA or CBF (*art. 11*) the angle BEC, will be equal to the angle DBA or DBF. Hence EC is parallel to BD, by *def. of parallel lines*, and $AD : DC :: AB : BE$ or BC, by *art. 67*.

T H E O R E M X I X.

78. If two triangles ABC, DEF, have one angle A in the one equal to one angle D in the other, they are to each other as the rectangles made of the sides about these equal angles. Fig. 6, 7.

Let AI, DL, be perpendicular to the bases AC, DF, and equal to the sides AB, DE; compleat the rectangles

D

gles

angles AK, DM and join the diagonals IC, LF: then if BG, EH, are perpendicular to AC, DF, the triangles ABC, AIC, having the same base AC, are as their altitudes BG, AI (*art. 66*) and the triangles DEF, DLF, having the same base DF, are likewise as their altitudes EH, DL. Hence $ABC : AIC :: BG : AI = AB$ by const. and $DEF : DLF :: EH : DL = DE$. But the right angled triangles AGB, DHE, having the angles A, D, equal by supposition, are equiangular (*art. 13*) and $BG : AB :: EH : DE$, (*art. 69*) therefore $ABC : AIC :: DEF : DLF$, by equality of ratio's (*art. 53*) or alternately (*art. 57*) $ABC : DEF (:: AIC : DLF) :: AK : DM$, by axiom 4.

C O R I.

79. Fig. 10. Since parallelograms AC, EG, are double the triangles ABD, EFH, of the same base and altitude, and the wholes are in the same ratio as their halves (*ax. 4*); equiangular parallelograms are to each other as the rectangles made of their sides about the equal angles.

C O R. II.

80. Hence, parallelograms or triangles, which have one angle A, in the one equal to one angle E in the other, and the sides about these equal angles reciprocally proportional, *viz.* $AD : EH :: EF : AB$, are equal: for they are as the rectangles made by the extremes AD, AB, and the means EH, EF, which being equal (*art. 73*) the parallelograms or triangles will be so to.

C O R. III.

81. On the contrary, two equal triangles or parallelograms, which have one angle in the one, equal to one angle in the other, the sides about these equal angles, are reciprocally proportional.

THEO.

THEOREM XX.

82. *Equiangular triangles ABC, DEF, are to each other in a duplicate ratio to that of their homologous sides.*

Fig. 8, 9.

If the angle A be equal to the angle D, and $AC:DF::AB:DE$: in AC, take AG a third continued proportional to AC and DF, and draw BG: then because $(AC:DF::)DF:AG::AB:DE$; the triangles ABG, DEF, which have the sides about the equal angles A, D, reciprocally proportional, are equal (80); and the triangles ABC, ABG, having the same altitude, are as their bases (art. 65) the triangle ABC, is therefore to the triangle ABG, or its equal DEF, as AC to AG; that is, in a duplicate ratio to that of their homologous sides AC, DF, by def. 8 of proportion.

C O R.

83. Fig. 10. Since similar parallelograms AC, EG, are double the similar triangles ABD, EFH, of the same bases and altitudes, and the wholes are in the same ratio as their halves (ax. 4) similar parallelograms are in the duplicate ratio to that of their homologous sides.

THEOREM XXI.

84. *All similar rectilineal figures, are in a duplicate ratio to that of their homologous sides.* Fig. 11.

Let all the angles of the figure ABCDE, be equal to all the angles of the figure KFGHI, respectively in the order they stand, and the sides between the equal angles proportional; by drawing lines from the equal angles C, G, to the equal ones A, K, and E, I; the triangles ABC, KFG, having the sides about the equal angles B, F, proportional, are similar (art. 70) and so the angles BAC, FKG, are equal, as well as their differences

ferences CAE, GKI, to the equal ones BAE, FKI, by ax. 2.

The similar triangles ABC, KFG, give $AB : KF :: AC : KG$; and as $AB : KF :: AE : KI$, by supposition, we get $AC : KG :: AE : KI$, by equality of ratios; and as the angles CAE, GKI, between these proportional sides are equal, the triangles CAE, GKI, are similar (art. 70); the triangles CED, GIH, having also the sides about the equal angles D, H, proportional by supposition, are likewise similar. Now as the similar triangles ABC, KFG, are in a duplicate ratio (art. 82) to that of their homologous sides AB, KF, or of AE, KI, as well as the similar triangles ACE, KGI; and the similar triangles EDC, IHG, in the duplicate ratio of ED, IH, or of AE, KI. The whole figure ABCDE is consequently to the whole figure FGHK, in the duplicate ratio of their homologous sides AE, KI, by art. 54.

Whatever the number of the sides may be, the figures may always be divided into as many similar triangles wanting two, and the wholes being in the same ratio as their like parts (art. 54); all similar rectilineal figures whatsoever are in a duplicate ratio to that of their homologous sides.

C O R. I.

85. Hence, all similar described figures with proportional lines are similar, and therefore in a duplicate ratio to that of their homologous sides.

C O R. II.

86. All figures, which are in a duplicate ratio to that of their homologous sides, are similar.

The following proportions are often used in mathematics, but may be passed over till they are wanted.

T H E O R E M XXII.

87. Rectangles made of proportional lines are proportional; if $a : b :: c : d$, and $e : f :: g : h$: I say, $ae : bf :: eg : dh$.

Because

Because $a : b :: c : d$, we have alternately $a : c :: b : d$, and $ae : ce :: bf : df$ (art. 65); or $ae : bf :: ce : df$; and because $e : f :: g : h$, we have alternately $e : g :: f : h$, or $ce : cg :: df : dh$, by the same article, or $ce : df :: cg : dh$; and therefore $ae : bf :: cg : dh$, by equality of ratios.

C O R. I.

88. Hence, if four lines a, b, c, d , are proportional, the squares made of these lines will be proportional. For if in the last article, the lines e, f, g, h , are supposed equal to the lines a, b, c, d , respectively, we get $aa : bb :: ce : dd$.

C O R. II.

89. If six lines a, b, c, d, e, f , are proportional, two by two; the square of any antecedent a is to the square of its consequent b , as the rectangle made of the remaining antecedents is to the rectangle made of the remaining consequents. For since $a : b :: c : d$, and $a : b :: e : f$, by supposition, we have $aa : bb :: ce : df$, by art. 87.

C O R. III.

90. If four rectangles or squares are proportional as well as one of their sides, the others will be proportional: for if $ae : bf :: cg : dh$, and $a : b :: c : d$; then will $e : f :: g : h$: if not, let $e : f :: g : x$; then will $ae : bf :: cg : dx$ (art. 87) and $ae : bf :: cg : dh$, by supposition; hence dx equal to dh (52) or $x = h$.

T H E O R E M XXIII.

91. If there be two ranks a, b, c , and d, e, f , of three continued proportionals, and the duplicate ratio of the two first in the one be equal to the duplicate ratio of the two first in the other, their single ratios will be equal.

For $\because a : b : c$, and $\because d : e : f$, by supposition, and $aa : bb :: a : c$, as likewise $dd : ee :: d : f$, (art. 84) and since $a : c :: d : f$, by supposition, we get $aa : bb :: dd : ee$, by equality of ratios, and $a : b :: d : e$, by art. 90.

C O R.

C O R.

92. Hence, if the single ratio of the first and second term, of two ranks of three continued proportionals, are equal, their duplicate ratios will be equal: that is, if in $\frac{a}{b}::\frac{c}{d}$, $\frac{b}{c}::\frac{e}{f}$, and $a:b::d:e$, then will $a:c::d:f$.

T H E O R E M XXIV.

93. If there be two ranks, a, b, c, d , and e, b, c, f , of four proportionals, two by two; in which the means in the one, are equal to the means in the other, or their extremes; the unequal ones will be reciprocally proportional.

For when four lines are proportional, the rectangle of the means, is equal to the rectangle of the extremes (art. 73) and the rectangle of the means, or extremes, being the same in both, the other rectangles of the unequal terms, will be equal, viz. $ad=ef$, and therefore $a:e::f:d$, by art. 75.

C O R.

94. Hence, if in two ranks of three continued proportionals a, b, c , and d, b, f ; the mean in the one be equal to the mean in the other, the extremes in the one, will be reciprocally proportional to the extremes in the other, viz. $a:d::e:c$.

T H E O R E M XXV.

95. If there be any number of four proportionals, in which the antecedent of the first ratio, be always equal to the consequent of the first ratio in the preceding rank of proportionals; the first antecedent, will be to the last consequent, in the compound ratio of all the remaining antecedents, to all the remaining consequents.

For if $a:b::c:d$, $b:f::g:h$; and $f:k::l:n$; then the first and second give $a:f::cg:db$; this, and the third $a:k::cgl:dbn$, by art. 87. The same thing will be true, if there are ever so many ranks of four proportionals. This explains the 9th definition of compound ratio.

N. B. Dele the two first lines in the next Page.

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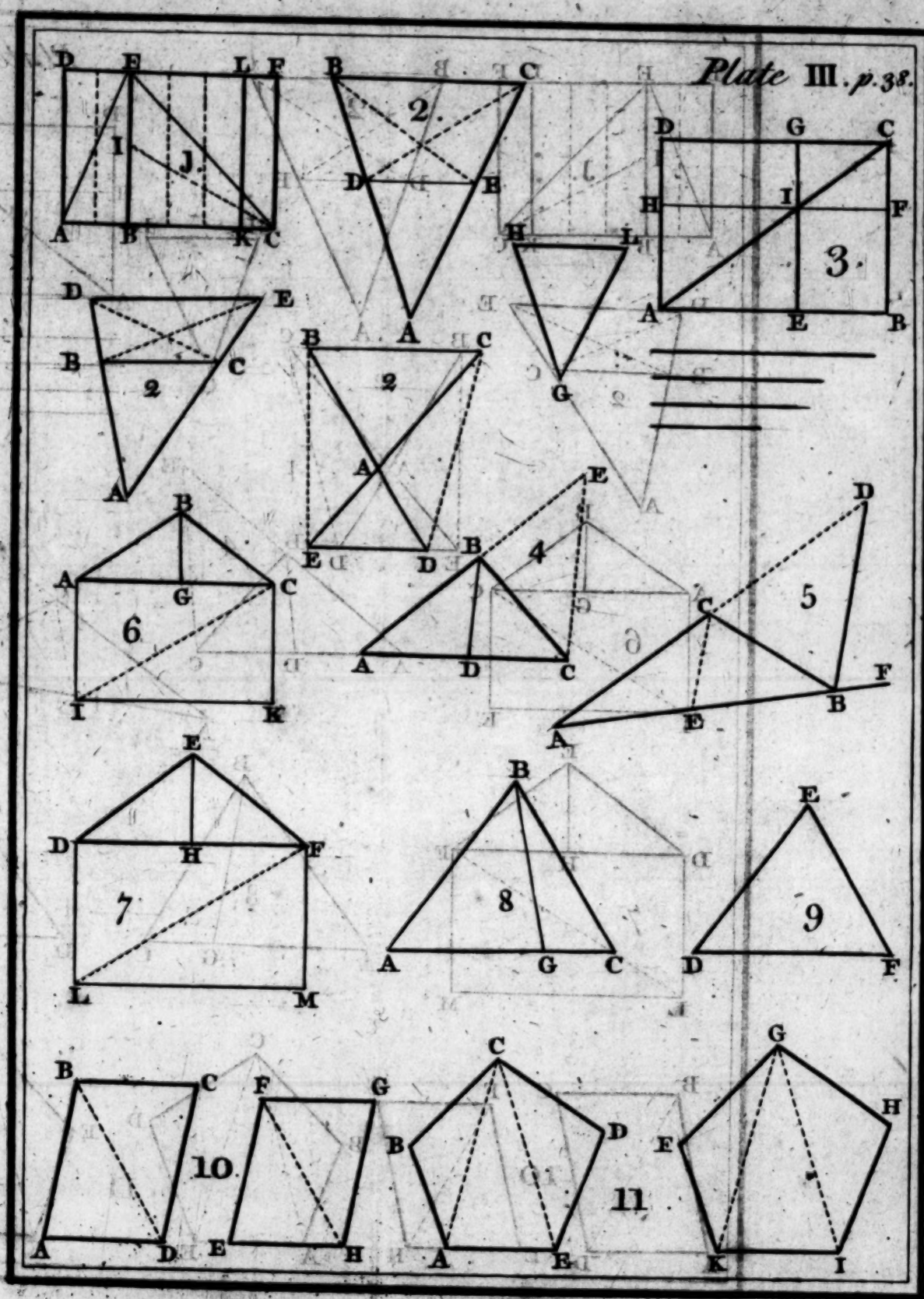
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SECTION V.

Of the CIRCLE.

DEFINITIONS.

1. **A** Circle is a plane figure bound by one continued curve line called Circumference, being every where equally distant from a point O within called the Center. Plate IV. Fig. 1.
2. A line O A, drawn from the center to the circumference is called a Radius.
3. Any line as A B, terminated by the circumference is called a Chord.
4. A chord divides the circle into two parts A E B, A C D B, call Segments.
5. The chord C D which passes through the center, is called Diameter.
6. Any part A E B, of the circumference is called an Arc.
Any part A O B E, of the circle, terminated by two radii and an arc, is called a Sector.
7. A line which only touches a circle in one point, and being produced both ways, falls without the circle, is called a Tangent.
8. The Measure of an angle is the arc, described from the angular point as center, with any radius, and terminated by the lines which form the angle.

C O R.

96. It follows from the definition of the circle, that all the *radii* of the same circle are equal, as likewise all the diameters (*ax. 6.*) which are each double the radius.

Grant that from a given point in a plane, with a given radius, or opening of the compass, the circumference of a circle may be described.

THEOREM I.

97. *In the same circle, equal chords AB, CD, subtend equal arcs; and the chords of equal arcs are equal.* Fig. 2.

First, Let the segment CED be placed on the segment AFB, so as the chord CD agrees with its equal AB, then the arc CED will agree with the arc AFB; for if the equal chords BF and DE are drawn, as well as the radii OB, OE, OD, OF, to their extremities, the triangles OBF, ODE, having all their sides equal, are equal in (*art. 28.*) all respects. Therefore the point E will agree with the point F; and as this will happen with regard to any points equally distant from the points B and D, it follows, that the arc CED agrees every where with the arc AFB. Hence equal chords subtend equal arcs.

Secondly, Since every part of the arc CED, agrees with its equal AFB, when the point C agrees with the point A, the point D will agree with the point B; the chord CD will agree with AB; and the chords of equal arcs are equal.

COR. I.

98. Hence, the radii drawn to the extremities of equal arcs, in the same circle, include equal angles. For if the arcs AFB, and CED, are equal, the chords AB, CD, are (*art. 97.*) equal also; and the triangles AOB, COD, having all their sides respectively equal, (*art. 28.*) are equal in all respects; and the angles AOB, COD opposite to the equal chords, are equal.

COR. II.

99. Hence also, the radii which include equal angles in the same circle, terminate equal arcs. For if the angles AOB, and COD, are equal; the triangles AOB,

AOB, COD, having the sides which include these equal angles (*art.* 96.) equal, are equal in (*art.* 15.) all respects. Therefore the chords AB, CD, opposite to the equal angles, are equal, as well as the (*art.* 97.) arcs AFB, CED, which they subtend.

C O R. III.

100. Fig. 3. Lastly, If two diameters AB, CD, are at right angles to each other, they will divide the circumference as well as the circle into four equal parts, called Quadrants. For the arcs and the opposite angles being equal, the sectors will be so too.

T H E O R E M II.

101. *The angles at the center in the same circle are proportional to the arcs terminated by the radii, which include these angles, viz.* $AB:BC::AOB:BOC$ Fig. 4.

Let the arc BL be any parts, of the arc AB; draw the radius OL; then because equal arcs subtend equal (*art.* 99.) angles, the angle BOL will be the same parts of the angle AOB, as the arc BL is of AB.

Now if the arc BL, is less than the arc BC, it is plain that the angle BOL is also less than the angle BOC; if the arc BL is equal to the arc BC, the angle BOL is equal to the angle BOC; and if the arc BL is greater than the arc BC, the angle BOL is greater than the angle BOC.

Since then, the equal parts BL, BOL of the antecedents AB, AOB, when compared with the consequents BC, BOC, are both together less, equal to, or greater than the consequents, the angles are proportional to the arcs which subtend them, by *defin.* 5, of proportion.

C O R. I.

102. The sectors AOB, BOC, are also in the same proportion to each other as the arcs AB and BC, which terminate them; for it is easy to perceive that equal arcs terminate equal sectors; and therefore, by the same argument as above, they are proportional to the said arcs,

N. B. Mathematicians suppose the circumference divided into 360 equal parts called Degrees, each degree into 60 equal parts called Minutes, and each minute into 60 Seconds. These divisions are commonly marked on a brass semi-circle, or on a rectangular piece of Ivory called Protractor, by which angles are measured or laid down on paper.

C O R. II.

103. Hence, the semi-circumference or the measure of two right angles is 180 degrees; the quadrant or measure of a right angle 90; and lastly, each angle of an equilateral triangle 60; as being the third part of two right angles.

T H E O R E M III.

104. *If a diameter KL bisects any chord AB, it will cut it at right angles; and if a diameter is at right angles to a chord, it will bisect it.* Fig. 5.

First, Let H be the intersection; draw the radii OA, OB, then the triangles OAH, OBH, having the sides OA, OB equal (*art.* 96.) by the nature of the circle, OH common to both, and HA equal to HB, by supposition are (*art.* 28.) equal in all respects; therefore the angles OHA, OHB, opposite to the equal sides OA, OB, are equal, and so are (*def.* 10.) right angles.

Secondly, If the angles OHA, OHB, are supposed to be right angles; then the right angled triangles OHA, OHB, having the hypotenuses OA, OB, equal by the property of the circle, (*art.* 96.) and the side OH common to both, are equal (*art.* 21.) in all respects; and the sides AH, HB, opposite to the equal angles at O, are equal.

C O R. I.

105. The line KL, which bisects any chord AB at right angles, passes through the center O of the circle; or if a line is drawn from the center perpendicular to any chord, it will bisect that chord.

C O R.

C O R. II.

106. If a line drawn through the center perpendicular to a chord be produced to the circumference, it will bisect the arc terminated by that chord; for the angles AOL and BOL being equal, the arcs AL , and BL , as likewise their differences AK , BK , to a semi-circle, are (*art. 99.*) equal.

C O R. III.

107. The arcs AC , BD , terminated by two parallel chords AB , CD , are equal. For the diameter KL , which bisects one of these chords AB , also bisects (*art. 7.*) the other CD , which is parallel to it, as well as the arcs terminated (*art. 106.*) by these chords; therefore the arcs AC , BD , which are the differences of equal arcs (*ax. 2.*) are equal.

T H E O R E M IV.

108. *The chords AB , CD , equally distant from the center O , are equal; and if the chords are equal their distances will be so.* Fig. 6.

Draw the radii, OA , OC , and the perpendiculars OH , OL , to the chords: then because OL , OH , are equal, by supposition, and the hypotenuses OA , OC , by the property of the circle, the triangles OHA , OLC , are equal in (*art. 21.*) all respects; therefore the sides AH , CL , or their (*art. 104.*) doubles AB , CD , are equal.

If the chords AB , CD , are equal, their halves AH , CL are (*ax. 6.*) equal, and the triangles OHA , OLC , right angled at H , L , having the sides LC , HA , and OA , OC , equal, are (*art. 21.*) equal in all respects: and the sides OH , OL , opposite to the equal angles OAH , OCL , are equal.

T H E O R E M V.

109. *The angle at the center COD is double the angle CAD at the circumference, which insists on the same arc CBD .* Fig. 7.

Draw the diameter AB and the radii OC , OD ; then, because OA , OC , are equal, the triangle AOC

is

is isosceles, and the external angle $\angle BOC$ is (*art.* 11.) double the internal opposite one $\angle BAC$; and since OA , OD , are equal, the triangle OAD is isosceles; and the external angle $\angle BOD$ double the internal opposite one $\angle BAD$. The external angle $\angle COD$, or the angle at the center, is therefore double (*ax.* 2.) $\angle CAD$ the internal opposite one, or the angle at the circumference, which insists on the same arc.

C O R. I.

110. Since the angle at the center is measured (*art.* 101.) by the arc CBD , its half, $\angle CAD$, will be measured by half that arc. Any angle at the circumference is then measured by half the arc on which it insists.

C O R. II.

111. If the chord AC produced be supposed to turn in the plane of the circle, about the point A as center, till the intersection C coincides with the point A ; it is evident, that this line will coincide with the tangent EF at A , since it touches the circle but in one point: and as the angle $\angle BAC$ is always measured by half the arc BC , so will the angle $\angle BAE$ be measured by half the circumference BCA . The tangent makes, therefore, a right angle with the diameter drawn thro' the point of contact A .

C O R. III.

112. As the angle $\angle BAC$ is measured by half the arc BC , and the angle $\angle BAE$ by half the arc BCA , the difference $\angle CAE$ between these angles will be measured by half the difference CA , of these arcs. Hence, any angle $\angle CAE$, made by a chord and a tangent, is measured by half the arc terminated by that chord, and so is equal to any angle at the circumference which insists on the same arc.

C O R. IV.

113. Fig. 8. The angles $\angle ACB$, $\angle ADB$, which are contained in the same segment, or, which is the same, insist on the same arc AEB , are equal; as being measured by half (*art.* 110.) the same arc AEB .

C O R.

C O R. V.

114. Each of the angles ACB , ADB , is less, equal to, or greater than a right angle, according as the segment in which they are contained is greater, equal to, or less than a semicircle; since half the arc AEB , which is their measure, will be less, equal to, or greater than a quadrant.

T H E O R E M IV.

115. *The angle contained between two lines drawn from any point E , to the circumference, is measured by half the sum or difference of the two arcs terminated by these lines, according as this point is within or without the circle.*
Fig. 9, 10.

Let A , B , C , D , be the intersections of these lines and the circumference, draw the chord AL parallel to CD , then the angle BED is (*def. 12.*) equal to the angle BAL , and the arc AC (*art. 107.*) equal to the arc LD . Now as the angle BAL at the circumference is measured by (*art. 110.*) half the arc LB , on which it stands, and the arc LB being the sum of the arcs BD , and AC , when the point E is within the circle, or their difference, when that point is without: it follows that the angle BED is measured by half the sum, or half the difference of the arcs BD , AC , according as the point E is within or without the circle.

T H E O R E M VII.

116. *If two lines AB , CD , intersect each other, either within or without the circle, their parts, terminated by the circumference, and the intersection, will be reciprocally proportional.*

Let the lines BC , AD be drawn, then the triangles EAD , ECB , having the angles at B and D equal, as insisting on the same arc (*art. 113.*) AC , and the angle at E common to both, are (*art. 69.*) similar: Therefore $AE : CE :: ED : EB$.

C O R. I.

117. If one of these lines passes through the center, and the other is perpendicular to it, as in Fig. 5, the diameter

diameter KL , to AB ; then $KH : AH :: HB : HL$, or because AH is equal to HL any perpendicular to a diameter is a mean proportional, between the parts of that diameter, terminated by the perpendicular.

C O R. II.

118. Fig. 11. If one of these lines becomes a tangent as EA , the triangles EAD , ECA , will still be similar; since the angles CAE , CDA , are both measured (*art. 112.*) by half the arc AC , and the angle E common to both; and $EC : EA :: EA : ED$.

C O R. III.

119. If from the same point E , there be drawn another tangent EB , to the circle, then will EC , $EB :: EB : ED$; or $EB = EA$; which shews, that if from the same point without the circle there be drawn two tangents, the parts between the points of contact, and their intersection, will be equal.

T H E O R E M VIII.

120. *The greatest chord that can be drawn in a circle is that which passes through the center, and the least that is farthest distant from it.* Fig. 12, 13.

From the center O draw the radii OC , OD , to the extremities of any chord CD , not passing through the center; draw likewise the diameter AB ; then because the sum of the sides OC , OD , of the triangle OCD is equal to the diameter AB , by the nature (*art. 96.*) of the circle, and always greater than the (*art. 23.*) third side CD ; it follows that the diameter is the greatest chord that can be drawn in a circle.

Now as the line CD recedes from the center O , the angle COD becomes less; therefore the (*art. 24.*) least chord is that which is farthest distant from the center.

T H E O R E M IX.

121. *If from any point E , either within or without the circle, there be drawn lines to the circumference; that*

EB

EB which passes through the center will be the greatest of all, and the part , of that line on the other side of the center, the least of all.

Because the sum of the sides DO and OE is equal to EB , and always greater than the (*art. 23.*) other side ED : it follows, that the line EB , drawn from any point E , which passes through the center, and is terminated by the circle, is the greatest of all.

Since OC is equal to OA , the line AE is equal to the difference between the sides EO and OC , and always less than the third (*art. 23.*) side EC . Consequently, the line drawn from any point E , which passes through the center when produced, and is terminated by the circumference is the least of all.

T H E O R E M. X.

122. *There can but two equal lines be drawn from any point E , besides the center to the circumference.*

Let the arcs BD , BF , be equal, draw FG , DC , through the point E , and join BD , BF ; then because the arcs BD , BF , as well as DA , FA , are equal by supposition, the chords BD , BF , are (*art. 97.*) equal; and as the side EB is common to the triangles EBD , EBF , and the angles at B are (*art. 113.*) equal, they are equal in all (*art. 15.*) respects. Therefore the sides ED , EF , opposite to the equal angles at B , are equal.

But as there can be taken but two arcs BD , BF , from the point B , which are equal, so can there neither be made but two triangles, which have the common side EB , that are equal in all respects.

C O R. I.

123. *If more than two lines can be drawn from any point to the circumference of a circle which are equal, that point will be the center.*

C O R. II.

124. *Two circles can cut each other but in two points, because the radii in the same circle are equal; the circle described from the center E can cut the circle $BD A F$ but in two points; since there can be drawn*
but

but two lines from that point to the circumference, which are equal.

C O R. III.

125. It is manifest, that two circles will cut each other when the distance between their centers is less than the sum, and greater than the difference of their radii; for there may always be described two circles from the centers E and O, with the radii ED, OD, which shall intersect each other (*art. 23.*) in the point D.

T H E O R E M. XI.

126. *If from the point M, where a tangent touches the circle, a perpendicular be drawn to any diameter BA, the radius will be a mean proportional between the parts of that diameter, terminated by the center, that line, and the tangent, viz. $\therefore OP : OA : OT$. Fig. 14.*

Let the radius OM be drawn; then, because the angle OMT, made by the radius and the tangent, is (*art. 111.*) a right one, and the angle O common to both, the right angled triangles OPM, OMT, are (*art. 13.*) equiangular. Therefore (*art. 69.*) $OP : OM :: OM : OT$, or $\therefore OP : OA : OT$.

T H E O R E M XII.

127. *If in a diameter produced there be taken two points P, T, such as the radius be a mean proportional, between the distances OP, OT, of these points from the center; the lines drawn from them to any point M in the circumference, will be always in a constant ratio, viz. $AP : AT :: PM : TM$. Fig. 15.*

In the radius OM, take OQ = OP, and join AQ; then the triangles OQA, OPM, having the angle O common to both, and the adjacent sides equal, are (*art. 15.*) equal in all respects; that is, $AQ = PM$; and because OP or OQ : OA :: OM : OT, by supposition, the line AQ will be (*art. 68.*) parallel to TM, and therefore (*art. 67.*) (OQ : OA ::) QM or AP : AT :: AQ or PM : TM.

T H E.

THEOREM XIII.

128. If in a diameter produced there be taken two points, P, T, so as the radius be a mean proportional between the distances of these points from the center; and if from one of them a line DT be drawn at right angles to the diameter; the line MN, joining the points of contact of two tangents drawn from any point D in that line, will always pass through the other point P. Fig. 16, 17.

From the point D draw a line to the center O intersecting the line MN in Q; then, because (art. 119) DM is equal to DN, and OM to ON, the triangles OMD, OND, having the side OD common to both, are (art. 28) equal in all respects.

Hence, the angles MOQ, NOQ, are equal; and since OM, ON, are equal, the line OD (art. 18) bisects MN at right angles.

Now the right angled triangles OTD, OQP, having the angle at O common to both, are (art. 13, 69) similar; whence $OQ:OP::OT:OD$; but (art. 126) $OQ:OA::OA:OD$; consequently (art. 85) $OP:OA::OA:OT$; and since $OP:OA::OA:OT$, by supposition, the line MN passes through the point P.

THEOREM XIV.

129. If two circles intersect each other in two points, the line OC joining the centers, will bisect the common chord AB at right angles. Fig. 18.

Draw the radii CA, CB, and OA, OB, then the triangles OAC, OBC, having all their sides equal, are (art. 28) equal in all respects; whence the angles ACO, BCO, are equal; and since the sides CA, CB, are equal, the line CO will bisect (art. 18) the chord AB at right angles.

These are all the most material theorems contained in the first, third, fourth, fifth, and sixth books of Euclid's elements, which are useful in the different branches of the mathematics, as far as I ever could find:

the rest are only of use to demonstrate some others by, for which reason we have omitted them.

As the problems are properly the application of the theorems to geometrical figures, we thought it more convenient to treat them separately; that the reader may peruse as many of them as he thinks proper, and leave the rest till he wants them.

S E C T. VI.

OF GEOMETRICAL PROBLEMS.

P R O B L E M I.

130. *THROUGH a given point C, to draw a line CD parallel to a given line AB. Pl. V. Fig. 1.*

From the given point C describe an arc EG, so as to cut the given line in E; in AB take EF equal to CE, and from the point F, as center, describe another arc through the point E, meeting the former in G; then the line CG will be parallel to AB.

For, if the lines CE, CF, and FG are drawn, the triangles CEF, CGF, having CF common to both, and the other sides equal by construction, are equal in all (art. 28) respects. Hence, the alternate angles CFE, FCG, opposite to the equal sides CE, FG, are equal. Consequently, CD is (art. 6) parallel to AB.

P R O B L E M II.

131. *From a given point C, either within or without a given line AB, to draw a perpendicular to that line. Fig. 2.*

From the given point C, as center, describe an arc so as to meet the line AB in two points E, F; and from these points, as centers, with the same radius greater than half of EF, describe two arcs so as to meet in D; then the line CD will be perpendicular to AB.

For if FD, ED, FC, EC, are drawn, the triangles CED, CFD, having CD common to both, and the other sides equal by construction, are equal (art.

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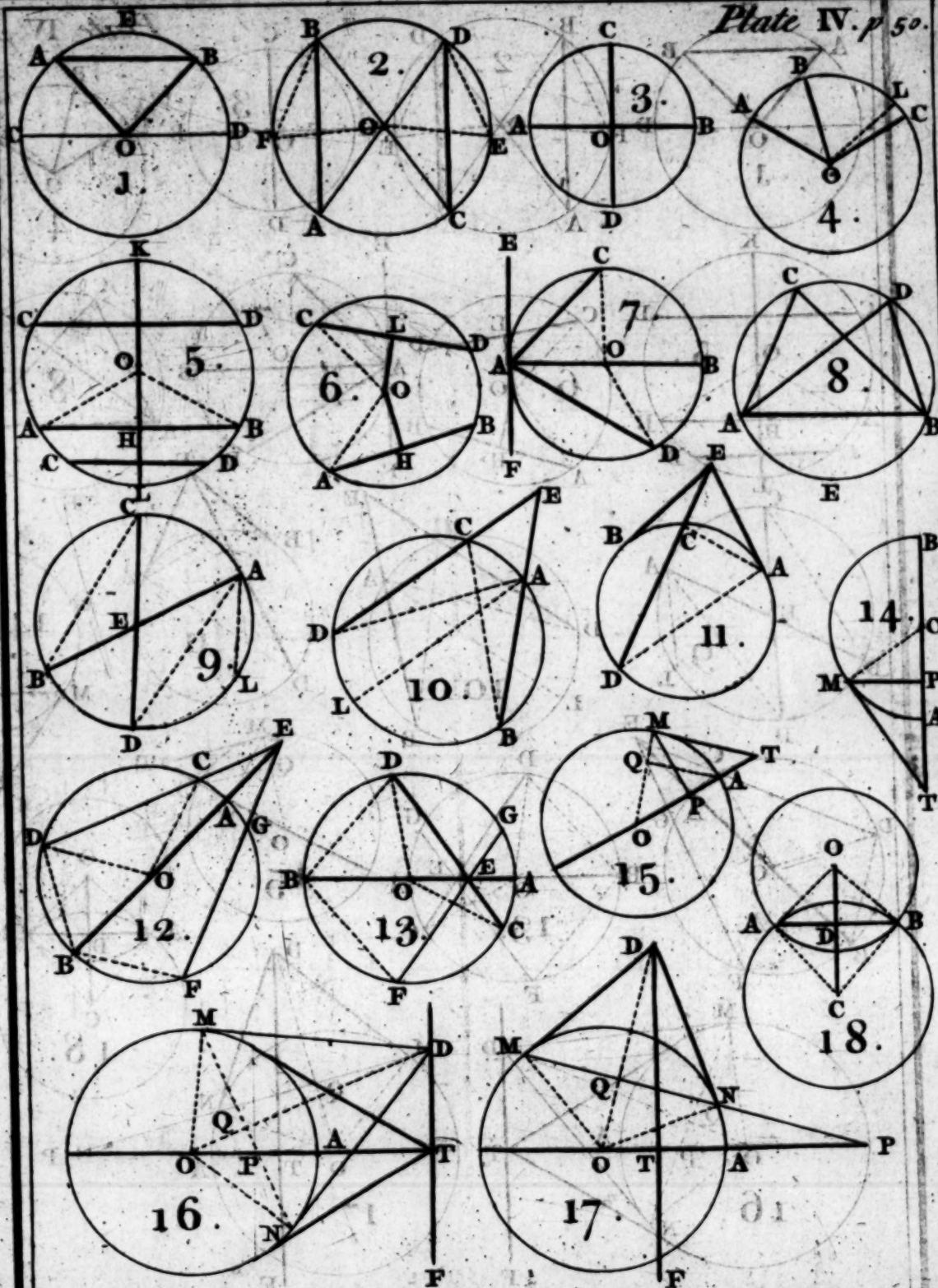
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(art. 28) in all respects; whence the angles EDC , FDC , are equal; and since ED , FD , (art. 18) are equal, the line DC is perpendicular to AB .

P R O B. III.

132. *At the extremity of a given line AB to erect a perpendicular BD to that line.* Fig. 3.

From any point O without the given line as center, describe an arc through the given point B , and so as to cut the line AB in some other point E ; draw the diameter ED ; then the line drawn through the extremity D to the given point B will be perpendicular to AB .

This is evident, since the angle EBD contained in a semi-circle is (art. 114) a right angle.

P R O B. IV.

133. *To bisect a given line AB by a perpendicular DE .* Fig. 4.

From the extremities A , B , of the given line as centers, describe arcs on both sides, with any radius, so as to intersect each other at D and E ; then the line DE drawn through these intersections, will bisect AB at right angles.

Let the points A , B , be joined to the points D , E ; then the triangles ADE , BDE , having DE common to both, and the other sides equal by construction, are (art. 28) equal in all respects: and the angles ADE , BDE , opposite to the equal sides AE , BE , are equal; since AD and BD are equal, the line DE (art. 18) bisects AB at right angles.

P R O B. V.

134. *To bisect any given angle ACB .* Fig. 5.

From the angular point C as center, with any radius, describe an arc meeting the sides in A , B , and from these points as centers, describe two arcs, with any radius, so as to meet in D ; then the line DC will bisect the angle ACB .

For since DC is common to both triangles, ACD , BCD , CA equal to CB , and BD to AD by construction,

struction, they are equal (*art.* 28) in all respects. And the angles ACD , BCD , opposite to the equal sides AD , BD , are equal.

P R O B. VI.

135. *With a given line AC to make an angle equal to a given one acb .* Fig. 6.

From the points C , c , as centers, describe two arcs with any radius; take the arc AB equal to the arc ab ; then the line CB will, with CA , make the angle ACB equal to the given one acb .

This is evident, since the radii CA , ca , are equal as well as the arcs AB , ab .

P R O B. VII.

136. *To describe a triangle ABC , whose sides shall be equal to three given lines AC , DE , FG , such that the sum of any two shall always be greater than the third.* Fig. 7.

From the extremity A of one of these lines as center, describe an arc, with a radius equal to DE , and from the other extremity C as center describe another arc with a radius equal to FG , so as to meet the first in B ; by drawing AB , CB , you will have the triangle required.

For since all the radii of the same circle are equal, the side AB is equal to the line DE , and the side CB to the line FG .

P R O B. VIII.

137. *To describe a rectangle AC , whose sides shall be equal to two given lines AD , EF .* Fig. 8.

At the extremities A , D , of one of these lines (*art.* 132) erect two perpendiculars AB , DC , each equal to the other line EF , and draw BC ; then will $ABCD$ be the rectangle, whose sides are equal to the given lines AD , EF .

This is evident from the construction; since AB , DC , are parallel and equal, BC will be parallel and equal to AD .

P R O B.

P R O B. IX.

138. To describe a parallelogram DF , equal to a given triangle ABC , whose sides shall make a given angle. Fig. 9.

Through the vertex B , of the given triangle, draw the line BF parallel to the base CA ; bisect (art. 133) the base AC in D , and draw DE so as to make with AC the given angle (art. 130); then, by drawing CF parallel to DE , the parallelogram DF will be equal to the given triangle ABC .

For since a triangle is half (art. 35) the parallelogram of the same base and altitude, the triangle ABC , whose base is double the base of the parallelogram DF , and has the same altitude, will consequently be equal to it; and the angle CDE is equal to the given angle by construction.

P R O B. X.

139. On a given line to describe a parallelogram EF , equal to a given triangle ABC , which shall have a given angle. Fig. 10.

Through the vertex B of the given triangle, draw a line BG parallel to the base AC , bisect the base in D , draw DI so as to make the given angle with DC ; produce DC to H so as CH be equal to the given line; draw CK , HF , parallel to DI , and through the point H and the intersection of BG and CK , draw HE till it meets DI in I ; then if IF be drawn parallel to BG , meeting CK , HF , in K and F , the parallelogram $EGFK$ will be the required figure.

For the parallelogram DE is (art. 138) equal to the triangle ABC , as likewise to its complement (art. 38) EF . Therefore the parallelogram EF is equal to the given triangle ABC , the side EG equal to the given line by construction, and the angle $G EK$ equal to the given angle.

P R O B. XI.

140. To make a rectangle GF , equal to any given trapezium $ABCD$. Fig. 11.

E 3

Bisect

Bisect the sides AB , BC , by the line EF , and the sides DA , DC , by GH ; and through the angles A , C , draw lines perpendicular to EF ; then will $EFHG$ be the rectangle required.

Draw the diagonal AC , then it is evident (*art.* 68) that EF , GH , are both parallel to AC , and so parallel to each other, whence GF is a rectangle; but the rectangle AF having the same base AC , and half the altitude of the triangle ABC , is (*art.* 35) equal to it; and the rectangle AH , having the same base, and half the altitude of the triangle ADC , they are likewise equal. Consequently, the rectangle GF is equal to the trapezium $ABCD$.

P R O B. XII.

141. *To make a triangle ABC , similar to a given triangle DEF , which shall have a given side AC .* Fig. 12.

At the extremity A of the given line, (*art.* 135) draw the line AB , so as the angle A be equal to the angle D , and at the other extremity C , the line CB , so as the angle C be equal to the angle F ; then the triangle ABC will be (*art.* 13) equiangular and (*art.* 69) similar to the triangle DEF , and the side AC equal to the given line.

C O R.

142. It is evident, that any rectilineal figure may be made similar to any other given figure, by reducing it into triangles, since there is no more required than to make upon a given line as many similar triangles as the given figure has been divided into; and therefore the triangles being similar in both, the figures themselves will be (*art.* 94) similar.

P R O B. XIII.

143. *To divide a given line AB into any number of equal parts, suppose 5.* Fig. 13.

Through the extremities A , B , draw the indefinite lines AC , BD , parallel to each other, making any angle with AB ; then if from the point A towards C , there

there be set off as many equal parts less one, on AC , as the line AB is to be divided into, and the same number of parts equal to the former, are set off on BD , from B towards D ; the lines which join the opposite points, will divide the given line into the desired number of equal parts.

For, because the lines $56, 67$, are equal and parallel to $12, 23$, the lines $15, 26$, are (*art. 31*) likewise parallel; therefore $A1 : 12 :: AE : EF$, &c. and since $A1$ is equal to 12 , AE will also be equal to EF , and whatever part $A1$ is of $A4$, AE will be the same part less one of AB .

P R O B XIV.

144. To find a fourth proportional to three given lines a, b, c . Fig. 14.

Draw AK and AL so as to make any angle, on AK take $AB = a$, $BC = b$, and on AL take $AD = c$; then if BD be drawn, and CE parallel to it, DE will be the fourth proportional required, since (*art. 67*) $AB : BC :: AD : DE$.

P R O B XV.

145. To find a third continued proportional to two given lines a, b .

On AK take $AB = a$, $BC = b$, and on AL take $AD = b$, join BD , and draw CE parallel to BD ; then will DE be the third proportional required. For $AB (a) : BC (b) :: AD (b) : DE$.

P R O B XVI.

146. To find a mean proportional between two given lines a, b . Fig. 15.

Draw the line AB equal to the sum of the two given lines; upon AB , as a diameter describe the semi-circumference of a circle AMB ; take AP equal to one of these lines a , and erect PM perpendicular to the diameter AB , which being terminated by the circumference, will be the mean proportional required by article 117.

P R O B. XVII.

147. To express the ratio of any two rectilineal figures P, Q , by two lines; that is, to find two lines which have the same ratio to each other as any two rectilineal figures. Fig. 16.

If they are parallelograms or rectangles, let $ABCD$ be equal to P ; make the parallelogram or rectangle DE equal (*art.* 139) to Q , whose side FE is given; then (*art.* 65) will $AC:DE::P:Q:AD:DF$.

If the figures P, Q , are trapeziums, you may find rectangles equal to them by *prob.* 11; and if they are any polygons, they may be reduced into triangles, and the triangles into rectangles.

P R O B. XVIII.

148. To describe the circumference of a circle through three given points, A, B, C , not placed in a right line. Plate VI. Fig. 17.

Join these points by two lines AB, AC ; which bisect (*art.* 133) by the perpendiculars DO, EO ; and from the intersection O of these perpendiculars, as center, describe the circumference through the point A , which will be the required one.

For since OD bisects the chord AB at right angles, the circumference described through the point A will (*art.* 105) pass through the point B ; and because OE bisects the chord AC at right angles, the circumference which passes through the point A will also pass through the point C . Therefore O is the center of the circle required.

P R O B. XIX.

149. From a given point T without a circle to draw a tangent to that circle. Fig. 18.

Join the center O to the given point T ; on OT , as diameter, describe a semi-circumference intersecting the former in M ; then the line drawn through this intersection to the point T , will be the tangent required. For if the radius OM be drawn, the angle OMT

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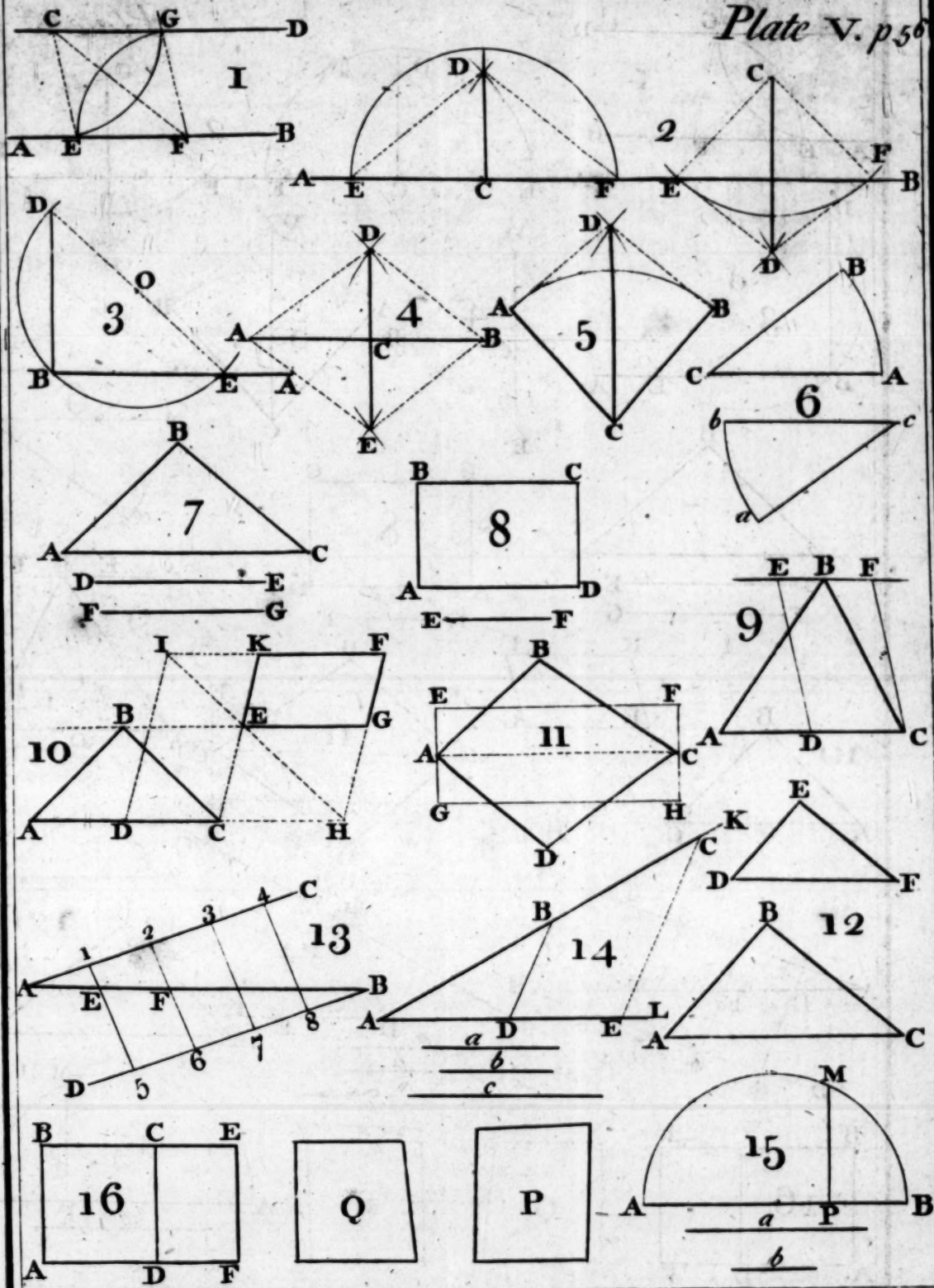
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contained in a semi-circle will be a (*art. 114*) right angle. Therefore the line MT touches the circle (*art. 111*), at the point M .

If it was required to draw a tangent from a given point M in the circumference, it is evident that the line, erected perpendicular at the extremity M , to the radius OM , will be the tangent required.

P R O B. XX.

150. To describe a triangle ABC in a circle, similar to a given triangle abc . Fig. 19.

Draw the tangent DE to any point B , and from the point B draw the chord BA , so (*art. 135*) as the angle DBA be equal to the angle c of the given triangle; and BC so as the angle EBC be equal to the angle a , join AC ; then will ABC be the triangle required.

For the angles ACB , ABD , are equal (*art. 112*) by the nature of the circle; and because the angle ABD has been made equal to the angle c , the angles ACB , c , (*ax. 1*) are equal; the angles BAC , CBE , are (*art. 112*) also equal: and since the angle CBE has been made equal to the angle a , the angles BAC , a , (*ax. 1*) are equal. Consequently, the triangle ABC , is (*art. 13*) equiangular, and (*art. 69*) similar to the triangle abc .

P R O B. XXI.

151. About a circle to describe a triangle DEF , similar to a given triangle abc . Fig. 20.

Produce the base ac of the given triangle to m , n ; draw the radii OA , OB , so as to make an angle equal to the external angle mab ; draw likewise the radius OC , so as the angle BOC be equal to the external angle ncb ; then the lines which touch the circle in A , B , C , will form the triangles DEF , similar to the triangle abc .

For the angles at A and B are (*art. 111*) right angles, and the angle AOB equal to the angle mab , by construction; the angle D , which makes two (*art. 32*) right angles with the angle AOB , will be equal to

to the angle $b a c$, which makes with the angle $m a b$, equal to the angle $A O B$, (*ax. 1*) likewise two right angles.

It is proved in the same manner, that the angle E , which makes two right angles with $B O C$, is equal to the angle $b c a$, which makes two right angles with its equal $n c b$. Therefore the triangle $D E F$ is equiangular (*art. 13*) and similar (*art. 69*) to the triangle $a b c$.

P R O B. XXII.

152. *To describe a circle which shall touch the three sides of a given triangle $A B C$. Fig. 21.*

Bisect (*art. 134*) any two angles B, C , by the lines $B O, C O$, and from their intersection O , draw (*art. 131*) $O D$ perpendicular to $A B$; then the circle described from the center O with the radius $O D$, will touch the three sides of the triangle $A B C$.

From the center O draw $O E, O F$, perpendicular to $B C$ and $A C$; then because the right angled triangles $B D O, B E O$, having the side $B O$ common to both, and the angles at B equal by construction, are (*art. 25*) equal in all respects; therefore $O D, O E$, opposite to the equal angles at B are equal.

And the right angled triangles $C E O, C F O$, having the side $C O$ common to both, and the angles at C equal by construction, are also (*art. 25*) equal in all respects; hence $O E$ and $O F$ are equal. Consequently, since the three lines $O D, O E, O F$, are equal and perpendicular to the sides of the triangle, the circle will touch (*art. 111*) the three sides of that triangle.

P R O B. XXIII.

153. *On a given line $A B$ as a chord to describe a segment of a circle, so as to contain a given angle. Fig. 22.*

Draw $D E$ such that the angle $D A B$ be equal to the given angle; let $A C$ be perpendicular to $A D$, and $B C$ to $A B$, meeting the former in C ; then the circle described on the diameter $A C$, will give the segment $A G C B$ required.

For

For CAD , ABC . are right angles by construction therefore the circumference (*art.* 114) passes thro' the point B , and touches the (*art.* 111) line DE ; and since the angle DAB has been made equal to the given angle, and is equal to any angle contained in the (*art.* 112) segment $AGCB$, it will be that required.

P R O B. XXIV.

154. *The angles AOB , COB , in which three objects A , B , C , placed in a right line, whose distances are known, are seen from any place O , being given; to find the distances from the place O to these objects.* Fig. 23.

Make the angles DCA , DAC , each equal (*art.* 135) to their alternate given angles AOB , COB ; thro' the point D and the extremes A , C , describe (*art.* 148) a circumference of a circle; then the line DB produced, so as to meet the circumference, will determine the point O .

For, the angles AOD , DCA , which insist on the same arc AD , (*art.* 113) are equal, as well as the angles BOC and DAC . Therefore the construction is rightly performed.

P R O B. XXV.

155. *The distances between three objects A , B , C , being given; to find a point D , such as the lines drawn from it to the three objects shall contain given angles* Fig. 24, 25.

On the side AB as a chord, describe a (*art.* 153) segment of a circle, so as to contain one of these given angles; describe likewise on AC as a chord, a segment so as to contain another of these given angles; then the intersection D of these arcs will be the point sought. This is evident by the construction.

N. B. When the point sought D is without the triangle; it is plain that if the angle ACB is less, equal to, or greater than the given angle ADB , the angle ABC must also be less, equal to, or greater than the given

given angle ADC , otherwise the problem is impossible; and when the angle BDC is equal to the sum of the angles ABC , BCA , the four points A, B, C, D , fall in the same circumference, and consequently the problem is indetermined in this case.

P R O B. XXVI.

156. *To describe the circumference of a circle through two given points C, G , so as the segment cut off by a line EF given in position, shall contain a given angle.* Fig. 26, 27.

Bisect in L the line joining the given points by the perpendicular OL , which meets the line EF in H , draw GH , and from the intersection L , the line LB , so as the angle LBH be equal to the difference between the given angle and a right one; then if LD is made equal to LB , and GO drawn parallel to DL , meeting HL in O , this point O will be the center of the circle required. For because OL bisects CG , at right angles, the circumference passing through one of these points will also pass (*art. 105*) through the other.

If OE be drawn parallel to LB , then (*art. 67*) since $HL:HO::LD:OG::LB:OE$; and as $LD = LB$, by construction, OE will be equal to OG ; and so the point E is in the circumference.

If EO be produced to the circumference K , then because any angle in the segment ECF is measured by half the arc EGF , and the angle KEF by half the arc FK , that is, by the difference between the arc EGF , and the semi-circumference EGK ; it follows that the angle OEH , or its equal LBH , is the difference between the given angle and a right angle.

N. B. First, If the line LB can be carried from the point L to any two different points in the line HG , produced either way, the problem will have two solutions; but if it cannot be carried to any point in that line it is impossible.

Secondly, If the given angle be a right one, the center O will coincide with the point H . For EF will be a diameter of the circle, as well as the line HL

pro-

produced; and therefore their intersection H will be the center.

Thirdly, If the line CG intersects the line EF at right angles, the line LO will be parallel to EF ; and hence $LO = BE$, and $LB = OE$; that is, LB will then be the radius. Therefore the two circular arcs described from the centers C, G , with the radius BL , will intersect each other in the center O .

Fig. 26. Fourthly. If the point L coincides with the point H ; that is, if EF bisects CG , then GH will coincide with LG ; whence, if from any point in HO , excepting the point L , the line LB be drawn as before, and carried from that point somewhere on CG , and the line GO parallel to this line will give the center O .

Fig. 27. Fifthly, And if C, G , coincide, so that the line CG becomes a tangent, and L the point of contact, then will HG become $= HL$, $OL = OG$, $HD = HL + (LD =) LB$. Therefore the proportion $HD:HG::LD:OG$, will become in this case, $HL + LB:HL::LB:LO$, by which the center O may be found.

When LB meets HG but in one point, it will be the shortest possible, and therefore the angle LBH or its equal OEH , will be the greatest possible. Now since this angle being the difference between the given angle and a right one; when the given angle exceeds a right angle, it will be the greatest possible; and when it is less, it will be the least possible.

C O R.

Fig. 28. 157. Hence, two points E, F , may be found in a right line given in position, from which lines being drawn to two other given points R, C , without that line, shall make a right angle ERF on one side, and any other given angle ECF on the other. For if RA be drawn perpendicular to EF , and in CA , produced, there be taken the point G , so as $CA:AR:AG$, the circumference of a circle described through

through the points C, G, and such that the segment ECF may contain the given angle, will intersect EF in the points required.

For by the property of the circle we have $AC : AE :: EA : AG$ and $CA : AR :: AR : AG$; hence (art. 116) $EA : AR :: AF$. Consequently, the angle ERF is a (art. 146) right angle.

This problem is useful in conic-sections, to find two conjugate diameters of an ellipsis or hyperbola, which shall make a given angle, when any two conjugate diameters are given.

S E C T I O N VII. R E G U L A R P O L Y G O N S. D E F I N I T I O N S.

1. **W**HEN a plane figure is terminated by more than four sides, it is called a Polygon.

2. When all the sides are equal as well as all the angles, the polygon is said to be Regular; and Irregular, if the sides or angles are unequal.

3. A polygon is said to be Inscribed in a circle, when all the angles touch, or are in the circumference.

4. And Circumscribed, when all its sides touch the circle.

5. The Angle at the Center of a polygon, is that which is made by two radii drawn to the extremities of the same side.

6. The Angle of the Polygon, is that which is made by any two contiguous sides.

7. Polygons are distinguished according to the number of their sides.

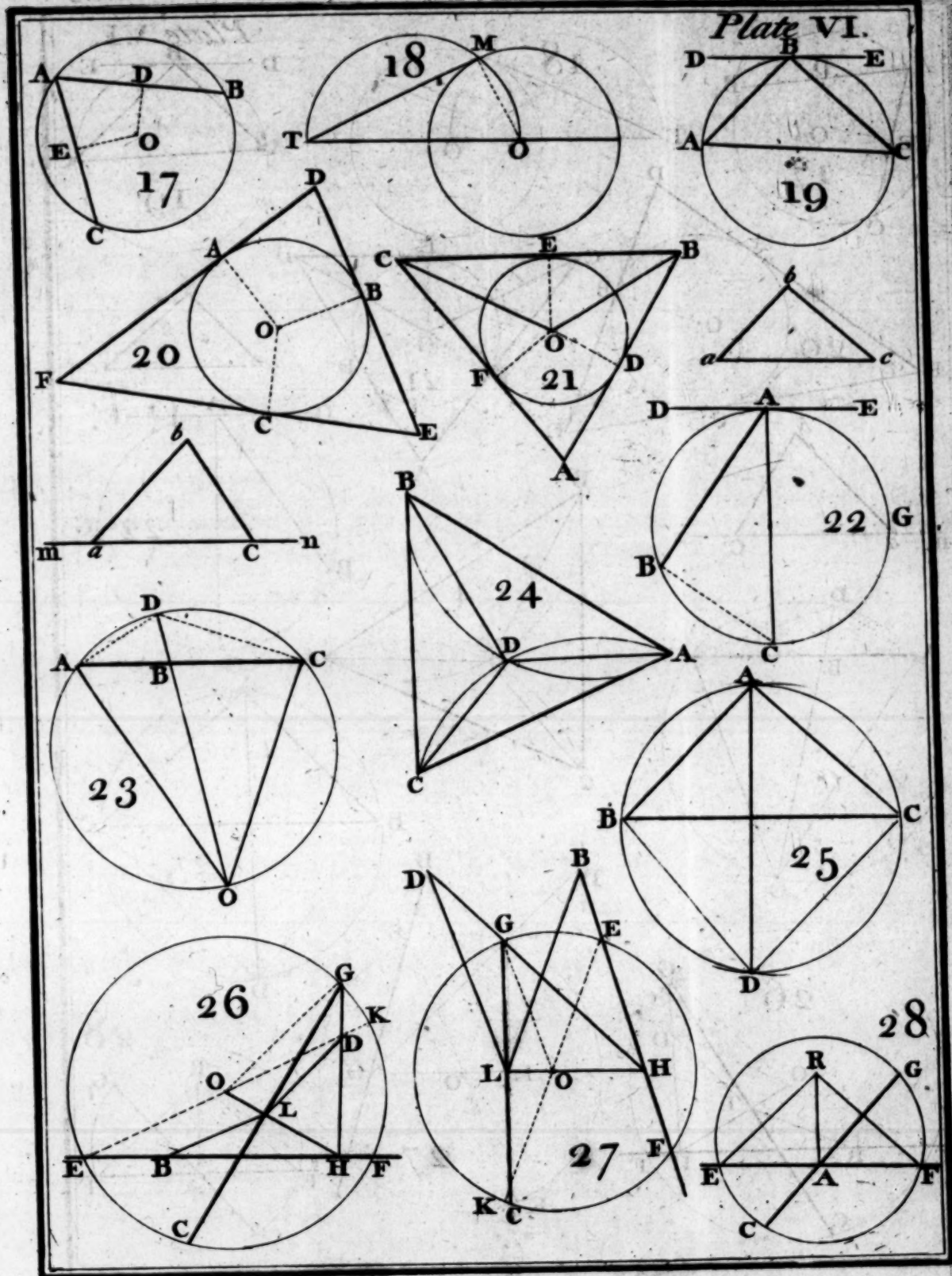
Number of sides,	{	5. Pentagon.		Number of sides,	{	9. Enneagon.
		6. Exagon.				10. Decagon.
		7. Eptagon.				11. Ondecagon.
		8. Octagon.				12. Dodecagon.

T H E O R E M I.

158. The angle AOB, at the center of a regular polygon, is equal to 360 degrees, or four right angles divided by the number of its sides. Plate VII. Fig. 1.

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For all the angles at the center taken together are equal to four (*art. 4*) right angles; that is, to 360 degrees, and their number to the number of the sides, because if radii are drawn to the extremities of all the sides, the polygon will contain as many equal isosceles triangles as there are sides: and therefore the sum 360 of all the angles, at the center divided by their number will be equal to the angle at the center required.

THEOREM II.

159. *The angle of a regular polygon is equal to the difference between two right angles and the angle at the center.*

For since AB and BC are equal by supposition, the isosceles triangles OAB , OBC , are (*art. 28*) equal in all respects: and hence the angles OAB , OBC , are equal. Therefore the angle ABC of the polygon, is equal to the sum of the angles OAB , OBA , of the triangle OAB ; and since the three angles of a triangle are (*art. 12*) equal to two right angles, the angle ABC of the polygon will be equal to the difference between two right angles, and the angle AOB at the center.

COR.

160. Hence, in any regular polygon, 360 divided by the number of the sides, gives the angle at the center of (*art. 158*) that polygon; and 180 multiplied by the number of the sides diminished by 2, and the product divided by the number of the sides, gives (*art. 159*) the angle of the polygon.

THEOREM III.

161. *If a Decagon be inscribed in a circle, the radius will be a mean proportional between the sum of the radius and the side of the decagon, and that side.* Fig. 2.

Let the arcs AB , BC , be each (36 degrees) the tenth part of the circumference; if the side AB , and the radius OC , both produced, meet in D . Then because

cause AB is the tenth part of the whole circumference, it will be the fifth of ACE ; and therefore the arc BCE four fifths, and half that arc, or the measure of the angle OAB , two fifths 72; that is, double the angle AOB at the center, or equal to the angle AOD ; therefore the triangle ADO is isosceles, and similar to the triangle (*art.* 13, 69) AOB . Hence, the angle ODA is equal to the angle AOB or BOD by supposition; so the triangle ODB , having two equal angles O, D , is likewise (*art.* 26) isosceles. Whence $BD=BO$, and AD is equal to the sum of the side AB of the decagon and the radius BO .

But because of the similar triangles ADO, OAB , we have (*art.* 69) $AB:AO::AO:AD$ or $AB+BO$. Consequently, the radius is a mean proportional between the sum of the side of the decagon and radius, and the side of the decagon.

THEOREM. IV.

162. *The sum of the squares of the radius and the side of a decagon inscribed in a circle, will be equal to the square of the side of a pentagon inscribed in the same circle.* Fig. 3.

Let AB be the side of the Pentagon, bisect the arc AB in C , then will BC or AC be the side of the decagon; bisect the arc AC in E , by the radius OE , and from its intersection D , with AB draw DC .

The arcs AC, CB , and AE, EC , are equal by construction, the arc EB will be 54 degrees, the three fourth of the arc ACB ; and therefore the angle EOB is (*art.* 160) the three fourths, 54 of 72, the angle AOB .

But the angle at the center AOB is (*art.* 158) two tenths, 72 of four right angles, or four tenths of two right angles; and as the three angles of a triangle AOB are equal (*art.* 12) to two right angles, each of the equal angles A or B will be three tenths of two right angles; that is, the angle OBA is the three fourths, 54, of the angle AOB , and so equal to the angle EOB . Consequently, the triangle OBD (*art.* 26) is isosceles and similar to the triangle AOB , as

having

having the common angle (*art.* 20) B. So then (*art.* 69) $AB:BO::BO:BD$, or (*art.* 73) the rectangle made by AB and BD is equal to the square of BO.

The radius OE bisects the arc AC, it also (*art.* 106) bisects its chord AC at right angles; and the lines DA and DC (*art.* 15) are equal. Whence the isosceles triangles ADC, ACB, having the angle A common to both (*art.* 20) are similar, and (*art.* 69) $AB:AC::AC:AD$; or (*art.* 73) the rectangle made by AB and AD is equal to the square of AC; and since the rectangles made of a line and its parts are equal (*art.* 49) to the square of that line, the square of AB is equal to the squares of OB and AC.

P R O B. I.

163. *To inscribe a square in a given circle.* Fig. 4.

Draw a diameter AB and another CD at right angles to it; then the lines joining their extremities will form the square ACBD required.

For the sides being the chords of equal arcs (*art.* 97) are equal, and each of the angles being contained in a semi-circle will (*art.* 114) be a right angle.

If the arc AC be bisected in E, the chord AE or EC of half that arc, will be the side of an octagon inscribed in the same circle.

It may be observed, that whatever the number of the sides of a polygon inscribed in a circle may be, you may always find the side of a polygon double the number of the sides, but not triple, nor any other unequal part, at least not by common geometry.

P R O B. II.

164. *To inscribe a pentagon or a decagon in a given circle.* Fig. 5.

Draw the radius OD at right angles to the diameter AB; bisect the radius AO in C, and from the point C as center, describe an arc through the point D, meeting AB in E; then OE will be the side of a decagon, and the line drawn from D to E the side of a pentagon inscribed in the same circle.

F

Let

Let the arc ED be continued so as to meet the diameter BA , produced in F ; then because AC , CO , and CF , CE , are equal by construction, AF will be equal to OE , and FO to AE . But by the property of the circle we have (*art.* 117) FO or AE : $OD::OD:OE$. Since then the radius OD is a mean proportional between the sum of the radius AO , the line OE , and that line; OE will be the (*art.* 161) side of a decagon inscribed in that circle. And because of the right angled triangle DOE , the (*art.* 49) square of DE is equal to the sum of the squares of OD and OE ; hence the line DE (*art.* 162) is the side of a pentagon inscribed in that circle.

When a line is divided into two such parts, that the greatest is a mean proportional between the whole line and the least, that line is said to be divided into mean and extreme proportion.

Hence, the radius OB is divided into mean and extreme proportion in E . For since $AE:OB::OB:OE$, by subtracting the consequents from their antecedents, and comparing the differences with the consequents, we have $OE:OB::BE:OE$, or inversely, $OB:OE::OE:BE$.

P R O B. III.

165. *To inscribe an exagon in a given circle.* Fig. 6.

If the radius AO be carried round the circumference, you will have the exagon required.

For the angle at the center O is the sixth part of four right angles, or one third of two right angles; therefore each of the equal angles A , or B , will likewise be a third part of two right angles. Consequently, the triangle OAB is equilateral.

If the arc AB be bisected in E , the chord of AE or EB will be the side of a dodecagon, inscribed in that circle.

P R O B. IV.

166. *To describe a polygon about a given circle of any number of sides.* Fig. 7.

Find

Find the side AB of a polygon inscribed in the circle of the same number of sides, draw the radii OA , OB , through its extremities, then the tangent DE parallel to AB , terminated by these radii produced, will be one of the sides of the polygon required, and the rest may be found in the same manner; or else describe a circle passing through the points D , E , then the side already found being carried round this last circumference, will touch the inner circle, and so will be the polygon required. This is self evident.

P R O B. V.

167. To find the radius of a circle, in which a polygon whose side is given, may be inscribed. Fig. 8.

Describe any circle at pleasure, in which find the side AB of a polygon similar to the required one; in AB , produced if necessary, take AD equal to the given side; draw DC parallel to the radius BO ; then CD or CA will be the radius required.

For the triangle CAD is similar to (art. 67) the isosceles triangle OAB ; and therefore the angle ACD equal to the angle at the center AOB , of the polygon required, and CA or CD will be the radius sought.

These and other such problems are easily solved by means of a sector, on the inside edge of which are marked the lines of polygons, from four to twelve sides: When the sector is opened, till the interval from 6 to 6 be equal to the radius of any circle, the distances between the respective numbers expressing the number of sides of the polygons, will be the sides of these polygons inscribed in that circle; viz. from 7 to 7, the side of an eptagon; from 8 to 8, the side of an octagon, &c.

On the contrary, if the sector is so opened as the interval between the respective numbers be equal to the given side of the polygon, then the distance from 6 to 6 will be the radius of the circle, in which that polygon may be inscribed.

168. *Similar polygons, inscribed in, or circumscribed about, circles, are proportional to the squares of their radii or diameters.* Fig. 9.

Let the radii be drawn to the extremities of the sides, then each polygon will be divided into as many equal triangles as there are sides, and those in the one will be similar to those in the other, as being isosceles, (*art.* 20) and having the angles at the center equal. Therefore (*art.* 69) $AOB:HQI::AO^2:HQ^2$, and if n expresses the number of the sides, (*art.* 55) $nAOB:nHQI::AO^2:HQ^2$; that is, the polygon in the one is to the similar polygon in the other, as the square of the radius in the first to the square of the radius in the second; or as (*art.* 55) the square of the diameter in the first is to the square of the diameter in the second.

THEOREM VI.

169. *Similar inscribed or circumscribed polygons have the sum of their sides proportional to the radii or diameters.*

Let n express the number of their sides, then because of the similar triangles OAB, QHI , we have (*art.* 69) $AB:HI::AO:HQ$; or, (*art.* 55) $nAB:nHI::AO:HQ::(art. 55)AD:HL$.

The same thing may be proved in respect to the circumscribed polygons.

THEOREM VII.

170. *The areas of circles are proportional to the squares of their radii or diameters.*

Inscribe and circumscribe similar polygons; then because (*art.* 55) $AO^2:nAOB::HQ^2:nHQI$; and we have proved that (*art.* 55) equal parts were proportional to the wholes; it is evident, that similar inscribed, or circumscribed polygons, are equal parts of the squares of the radii or diameters.

Now because, if the parts $nAOB$ of the first antecedent AO^2 are less than the circle or its consequent; the parts $nHQI$ of the second antecedent HQ^2 , will also be less than the circle or its consequent: if the

parts

parts $nAOB$ are greater than its consequent; the parts $nHQI$ are also greater than the consequent; and if possible, that the parts $nAOB$ are equal to the circle, the parts $nHQI$ must also be equal to the circle; for if one was either greater or less, we have proved that the other must likewise be greater or less. Consequently, since the equal parts of the antecedents, when compared to their respective consequents, are both together less, equal to, or greater than the consequents, these quantities are proportional by defin. 5 of proportions.

THEOREM VIII.

171. *The circumferences of circles are proportional to their radii or diameters.*

Let n be the number of the sides of similar inscribed or circumscribed polygons; then as (*art. 55*) $AO : nAB :: HQ : nHI$, it is evident, by what has been said in the last proposition, that the sums nAB , nHI , of the sides of the inscribed or circumscribed polygons, are equal parts of the radii or diameters: Now it is plain, that if the parts nAB are less than the circumference ACE , the parts nHI are also less than the circumference HKM ; if the parts nAB are greater than ACE , the parts nHI are likewise greater than HKM , and if possible that nAB is equal to ACE , the parts nHI must also be equal to HKM : Since if one is either greater or less, the other is also greater or less. Therefore, since the equal parts of the antecedents, when compared with the respective consequents, are both together less, equal to, or greater than the consequents; these quantities are proportional by defin. 5 of proportion.

C O R.

172. Whatever has been demonstrated in regard to the whole circles, is likewise true in regard to their similar parts; that is, the sector AOB is to the similar sector HQI , as the square of AO to the square of HQ ; and any arc AB is to the similar arc HI , as AO to HQ . By similar arcs and sectors we understand those which

are terminated by radii that make equal angles at the center. This is evident, since the wholes are proportional to their equal parts.

THEOREM IX.

173. *The area of a circle is equal to a triangle OFL, whose base FL, is equal to the circumference, and altitude to the radius.*

As the area of any polygon is equal to a triangle, whose base is equal to the sum of its sides, and whose altitude, the line drawn from the center perpendicular to one of its sides; and since the area of any inscribed polygon is always less than that of the circle, as also less than the triangle OFL; and the area of any circumscribed polygon is always greater than that of the circle, as likewise greater than the triangle OFL: And because these polygons may approach so near to equality as no difference can be assigned, it is evident, that the circle and the triangle OFL, which are always greater than the one, and less than the other, must be equal.

COR. I.

174. Hence, the area of a circle is equal to half the rectangle made of the circumference and the radius, or to one fourth part of the rectangle made by the circumference and the diameter.

COR. II.

175. Since the area of the whole circle is equal to half the rectangle made of the circumference and the radius, it is manifest, that the area of any sector is equal to half the rectangle, made of the radius and the arc which terminates the sector.

As the seventh and eighth theorems have been immediately demonstrated from the definition of proportions, we presume that it is more agreeable to the strictness of geometrical reasonings, than any other method that has hitherto appeared.

SECT.

S E C T. VIII.

O F S O L I D S.

D E F I N I T I O N S.

1. **A** Sphere is a solid described by the rotation of a semi-circle about its diameter as an axis.
 2. A Cube is a solid contained between six equal squares.
 3. A Parallelepiped is a solid contained between six parallelograms, the opposite ones being parallel and equal.
 4. A Prism is a solid whose opposite bases are two equal and parallel polygons and the sides parallelograms.
 5. A Cylinder is a solid whose opposite bases are equal and parallel circles and the section thro' the centers of the opposite bases is always a parallelogram.
 6. A Pyramid is a solid whose base is any polygon, and terminates in a point, whose sides are triangles.
 7. A Cone is a solid whose base is a circle, terminating in a point in the same manner as a pyramid; and any section through the upper point and the center of the base is always a triangle.
 8. A Frustum of a pyramid or cone is the remainder of a greater from which a less has been cut off by a plane parallel to the base.
 9. The upper point of a pyramid or cone is called the Vertex.
 10. The line which joins the centers of the opposite bases in a prism, cylinder, or the vertex to the center of the base, in a cone or pyramid, is called the Axis.
 11. From whence solids are said to be right or oblique, according as the axis is perpendicular or oblique to the base.
- Therefore a right cylinder may be described by the rotation of a rectangle about one of its sides as an axis; and a right cone by the rotation of a right angled triangle about one of its sides as an axis.

12. Similar solids are those which are comprehended by an equal number of similar planes, placed similarly.

13. Similar cones or cylinders are those whose axes incline equally on their bases, and are proportional to the diameters of their bases.

14. The Side of a cone or cylinder, is the intersection of a plane passing through their axis, with their surface.

A X I O M.

Triangular prisms, which are comprehended between equal and similar planes, are equal; as filling up exactly the same space.

T H E O R E M I.

176. *Parallelepiped* $ACEM$, $ACGK$, which have the same or equal bases, $ABCD$, and the same altitude; or, which is the same, being terminated by the same plane $MEHI$, parallel to the base AC , are equal. Fig. 10.

Since CE , BH , are equal parallelograms, the triangles BEG , CFH , are (*art. 35*) equal, as well as their opposite ones AMK , DLI ; and because EG is equal to FH , the parallelograms $MEGK$, $LFHI$, having also the same height, are likewise equal; as well as $ABGK$, $DCHI$, and $ABEM$, $DCFL$, being the opposite sides of the same parallelepiped. Therefore the triangular prisms $ABEGKM$, $DCFHIL$, being terminated by the same number of equal and similar planes, are by axiom, equal; if these are taken from the same solid $AEHI$, we shall have the parallelepiped $ACEM$ equal to the parallelepiped $ACGK$. Consequently, parallelepipeds, which have equal bases and equal altitudes, are equal.

T H E O R E M II.

177. *If a parallelepiped* HB be cut by a plane $HDBF$, passing through the opposite angles, the two triangular prisms EDF , GBF , are equal. Fig. 11.

For the opposite planes AF , DG , and HA , GB , are parallel and equal by defin. the diagonals DB , HF , divide the parallelograms AC , EG , which are parallel and equal (*art. 31*) into equal and similar triangles,

and

and the plane DBFH being common to both, the prisms EDF, GBF, which are terminated by the same number of equal and similar planes, are similar and equal. Consequently, the diagonal plane HDBF, divides the parallelepiped into two equal and similar triangular prisms.

C O R. I.

178. Hence, all triangular prisms right or oblique, which have equal bases and equal altitudes, are equal; since their doubles are (*art. 54*) equal, they must be so too.

C O R. II.

179. Fig. 12. It is manifest, that all prisms HA whatsoever right or oblique, which have equal bases and equal altitudes, are equal; since they may be always divided into as many triangular prisms as their opposite bases ABCDE, FGHIK, can be divided into triangles; and the parts in the one being equal to the respective parts in the other, their sums or the whole prisms, will be equal.

T H E O R E M III.

180. *If a prism or cylinder be cut by a plane parallel to the base, the section will be similar and equal to the base.*

Fig. 12. *First*, Let LMNOP be a section parallel to the base; then because all the sides of the prism are parallelograms by defin. and KL, IM, NH, parallel and equal by supposition, $LM=KI$, $MN=IH$; but the angles L, M, N, O, are also equal to the respective angles K, I, H, G, as being formed by parallel lines in parallel planes; and as this will always happen let the plane pass where it will; and figures, which have all their sides and angles equal, are equal in all respects: it follows that the section LMNO, which is parallel to the base KIHG, is also equal to it.

Fig. 13. *Secondly*, Let ABCD be a section through the axis of the cylinder, AB, DC its sides; draw the diameters A-D, B-C, and from any point E draw EF parallel

parallel to AD or BC; then because ABCD, is always a parallelogram, by defin. BC and AE, DF, parallel and equal by supposition $EF = AD$: and as this always happens, wherever the line EF is drawn in any section of the cylinder, and the diameters of the bases being always the same, the section through FE, or through any other place, parallel to the base, will be always a circle, equal to the base.

T H E O R E M IV.

181. *If a pyramid or cone be cut by a plane parallel to the base, the section will always be similar to the base.* Plate VIII. Fig. 16.

First, Let EFGH be a section parallel to the base ABCD; then by reason of the parallelism, we (*art. 67*) have $(KF:KB::) FE:BA::FG:BC$, and $(KG:KC::) GF:CB::GH:CD$. Therefore, $EF:BA::FG:BC::GH:CD$, by equality (*art. 53*) of ratios; and since the angles E, F, G, H, and A, B, C, D, are formed by parallel lines in parallel planes respectively, they are equal. Consequently, the planes EFGH, ABCD, having all their angles equal, and their sides proportional are similar.

Fig. 17. *Secondly,* If from any point C in the axis of the cone, a line CD be drawn parallel to the radius AB of the base; then will (*art. 67*) $KC:KA::CD:AB$; and since all the radii of the base are equal, all the lines drawn from the same point C, in the axis, in a plane parallel to the base, will also be always equal. Consequently, that plane, or the section parallel to the base, will always be a circle.

T H E O R E M V.

182. *Parallelepipeds AD, FI, which have the same or equal bases ABC, are as their altitudes AF, FG. I say, $AF:FG::AD:FI$.* Fig. 14.

In FG take FL equal to any parts $\frac{1}{2}$ of AF, and let a plane LMN pass through the point L, parallel to the base GHIK; then because parallelepipeds, which have

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have equal bases and equal altitudes (*art. 176*) are equal, the parallelepiped FN will be the same parts, $\frac{1}{2}$, of the parallelepiped AD , as FL is of AF . Now it is evident, that if FL is less than FG , the parallelepiped FN is less than FI ; if FL is equal to FG , the parallelepiped FN is equal to FI ; and if FL is greater than FG , the parallelepiped FN is also greater than FI : since the equal parts FL , FN , of the antecedents AF , AD , when compared to the consequents FG , FI , are both together less, equal to, or greater than the respective consequents; these quantities are proportional by defin. 5 of proportions.

C O R. I.

183. Fig. 15. It is manifest that parallelepipeds AB , EL , which have equal altitudes DB , HL , are as their bases, AD , EH ; for with GD make the parallelogram CD equal to the base EH , complete the parallelepiped CB ; then as the parallelepipeds CB , EL , have their bases and altitudes equal by supposition (*art. 176*) they are equal; and the parallelepipeds AB , CB , having the same base GB are as their altitudes AG , GC ; or as (*art. 65*) AD , CD ; but the base CD has been made equal to the base EH ; and the parallelepiped EL is equal to CB . Consequently, $AB:EL::AD:EH$.

C O R. II.

184. Since every parallelepiped is equal to two triangular (*art. 177*) prisms, whose bases are each half that of the parallelepiped and of the same altitude, whatever is true in the whole will also be true (*art. 54*) in the halves. Consequently, triangular prisms, which have equal bases, are as their altitudes; and if their altitudes are equal, they are as their bases.

C O R. III.

185. As any prism whatsoever, right or oblique, may always be divided into as many triangular prisms as the base can be divided into triangles; and since whatever is true in respect to parts, is likewise true (*art. 54*) in the wholes, it is evident that any two prisms, which have

have equal bases, are as their altitudes; or, if their altitudes are equal, they are as their bases; or, if any of their dimensions are equal, they will always be as the unequal ones.

L E M M A.

186. *If a plane EH moves parallel to itself, and any two parts EF, GH, of that plane, are constantly equal to each other in any situation whatsoever: I say they will describe equal solids at the same time.* Fig. 18.

If it be denied that they are equal, they must be unequal in some parts; as the plane EH is parallel to the base AB, by supposition, the solids AEFB, AGHB, being terminated by parallel planes, have equal altitudes; they have also equal bases AB; if they are unequal, it must be somewhere between the planes AB, EH, but the parts of a plane passing through these solids between AB, EH, parallel to the base AB, are always equal by supposition. Since, therefore, no parts of the solids AEFB, AGHB, can be unequal, they must necessarily be equal. Consequently, solids described by equal parts of the same plane, moving parallel to itself, are equal.

C O R. I.

187. Hence, all cylinders right or oblique, which have the same or equal bases and are terminated by the same plane, parallel to the base, are equal; for we have shewn (*art. 180*) that any section parallel to the base is always a circle, equal to the base. Consequently, cylinders of equal bases and equal altitudes, are equal.

The same thing is likewise true in respect to any prisms which have equal bases and equal altitudes.

C O R. II.

188. It appears also, that a prism and a cylinder, which have equal bases and equal altitudes, are equal: For, since any sections in both, parallel to the base, are (*art. 180*) always equal to the base, the bases being equal the sections will be equal; and since the altitudes are likewise equal by supposition (*art. 179*) the solids them-

themselves will be equal. Consequently, a cylinder may be considered as a prism, and whatever is demonstrated in the one, will equally be true in the other.

C O R. III.

189. Since prisms which have equal bases are as their altitudes (*art.* 185) or if their altitudes are equal, they are as their bases: Cylinders which have equal bases and altitudes with prisms, being equal to them, will likewise be as their altitudes, if their bases are equal, or as their bases, if their altitudes are equal.

This property of prisms and cylinders may be deduced immediately from the definition of proportions, without having recourse to the article 185, since it entirely depends on the equality of solids which have equal bases and altitudes; but we have chosen to deduce them in this manner, in order to avoid the multiplicity of figures.

T H E O R E M VI.

190. *All cones or pyramids which have the same or equal bases, and equal altitudes, are equal.* Fig. 18.

Let the triangles ACB, ADB, represent the sections of two pyramids or cones, which have the same base AB and the same altitude, and let EH represent a section parallel to the base; then because of the parallels AB, EH, CD, we have (*art.* 67) $DG:DA::GH:AB$, and $DG:DA::CE:CA::EF:AB$; therefore, $GH:AB::EF:AB$, by (*art.* 53) equality of ratios: And hence, $GH=EF$. Now because EF and GH represent the diameters of (*art.* 181) circles, in the cones, and two lines in the pyramids, similarly placed; and since these lines are constantly equal, the circles in the cone, or the similar planes in the pyramids, are likewise always equal. Consequently, the cones or pyramids, described by these similar and equal planes, are (*art.* 186) also equal.

C O R.

191. It is manifest, that a cone is equal to a pyramid of an equal base and altitude; for since the ratio of the circle

circle of any section in the cone, to the circle of the base, is equal to the ratio of the section of a plane in the pyramid to the base, which are equally distant from the base, and the bases being equal, these sections will be equal: And consequently, the solids described by these equal planes will be (*art. 186*) equal; whence, whatever is proved of pyramids, will likewise be true in respect to cones.

THEOREM VII.

192. *A triangular prism ABCDEF is triple the pyramid BFED of the same base FED and the same altitude BE.* Fig. 19.

Let two planes pass through FBC, and DBF, then because the diagonal CF divides the parallelogram DCAF into (*art. 31*) two equal triangles CDF, CAF, the pyramids CAFB, CDFB, having equal bases CDF, and CAF, and the same vertex B, are (*art. 190*) equal; and the pyramids CAFB, DEFB, having the bases CBA, DEF, equal by defin. and the same altitude EB, are (*art. 190*) also equal. Since the pyramid CAFB is equal to both the pyramids CDFB, and DEFB, they are all three equal to each other. Hence, the prism ABCDEF is triple the pyramid BFED, of the same base and altitude.

COR. I.

193. Since any prism may be divided into triangular ones, and as a triangular pyramid is one third of the triangular prism of the same base and altitude; it follows, that any pyramid is equal to one third of the prism of the same base and altitude.

COR. II.

194. As a cone is equal to a pyramid of an (*art. 191*) equal base and altitude, and a cylinder equal to a prism of an (*art. 188*) equal base and altitude, it is manifest, that a cone is one third part of a cylinder of the same base and altitude.

COR. III.

195. Because prisms and cylinders, which have equal bases, are (*art. 185*) as their altitudes; or, if their altitudes

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altitudes are equal, they are as their bases; pyramids or cones, which have equal bases, are likewise as their altitudes; or, if their altitudes are equal, as their bases, since the wholes are proportional to (art. 54) their equal parts.

C O R. IV.

196. Since the pyramid DBFE is one third of the prism DCABFE, it follows, that the remainder DCBAFD will be the two thirds of that prism. This must be observed, because it serves hereafter to find the number of shot contained in a square or oblong Pile.

T H E O R E M VIII.

197. If three lines A, B, C, are in a continued proportion, the rectangular parallelepiped EL, made of these three lines, will be equal to the cube FK, made of the mean B. Fig. 15.

Let the base EH be equal to the rectangle of the extremes A, C, and the altitude HL equal to the mean B; then because $A:B:C$, the base EH is (art. 73) equal to the base FI; and since the altitudes HL, IK, are also equal by supposition, the parallelepipeds EL, FK, are equal.

T H E O R E M IX.

198. If four lines A, B, C, D, are in a continued proportion, the rectangular parallelepiped CB made of the square of the first A and the fourth D, will be equal to the cube FK of the second B.

Let CG be equal to the fourth D; GD, DB, equal to the first A; and let the base EH be equal to the rectangle of the second B and the third C; and the altitude HL equal to the first A; then because $A:B:C:D$, the base CD is (art. 72) equal to the base EH; and since the altitudes DB, HL, are equal by construction, the parallelepipeds CB, EL, which have their bases and altitudes equal, are (art. 176) equal; and as the parallelepiped EL is made of three lines A, B, C,

A, B, C, which are in a continued proportion, it will be equal (*art. 197*) to the cube FK, made of the mean B. Since, therefore, the parallelepiped EL is equal to both the cube FK and the parallelepiped CB, the cube FK will be equal to the parallelepiped CB.

T H E O R E M X.

199. *If four lines A, B, C, D, are in a continued proportion, the cube AB, of the first A, will be to the cube FK of the second B, as the first line A is to the fourth D.*

Let GC be equal to the fourth line D, and compleat the rectangular parallelepiped CB, on the base GB; then because the parallelepiped CB is made of the square GB of the first, and its altitude GC is equal to the last, it will be equal to the cube (*art. 198*) FK made of the second B; but AB, and CB, having the same base GB, are as their (*art. 182*) altitudes: Therefore, $AB:CB$, or $FK::AG:GC$, or as A to D.

C O R.

200. Hence, cubes are in a triplicate proportion to that of their sides; since, if four lines are in a continued proportion, the ratio of the first to the last is said to be triplicate to that (*def. 8 of proportion*) of the first and second.

T H E O R E M XI.

201. *If there are six lines A, B, C, D, E, F, proportional, the cube AB of any antecedent A, is to the cube FK, of its consequent B, as the rectangular parallelepiped CB, of the antecedents A, C, E, is to the rectangular parallelepiped EL of the consequents B, D, F.*

Let the base CD be equal to the rectangle of the second and third antecedents C, E, and the base EH equal to the rectangle of the second and third consequents D, F; compleat the parallelepipeds CB, EL, with the same altitudes of the cubes AB, FK; then the solids AB, CB, having the same altitudes, are (*art. 182*) as their bases AD, CD; and the solids FK, EL, having the same altitudes, are also as their bases FI, EH.

But

But when six lines are proportional, the square AD of the first is to the square FI of the second, (*art. 81*) as the rectangle CD of the second and third antecedents is to the rectangle EH of the second and third consequents: Therefore $AB:FK::CB:EL$. Consequently, if six lines are proportional, the cube AB of any antecedent A , is to the cube FK of its consequent B as the parallelepiped CB , of the antecedents, is to the parallelepiped EL of the consequents.

THEOREM.

202. *Rectangular parallelepipeds AB , FK , which have their bases and altitudes reciprocally proportional are equal; if $PK:GB::AG:PF$. I say, $AB=FK$.*

In AG produced, take GC equal to PF ; complete the solid CB , upon the base GB , and altitude CG ; then the solids FK , CB , having equal altitudes PF , GC , by construction, are as their bases; that is, $FK:CB::PK:GB$; and the solids AB , CB , having the same base GB , are as their altitudes; *viz.* $AB:CB::AG:GC$; but $PK:GB::AG:PF$, by supposition; and as GC , PF , are equal by construction, we have $FK:CB::AB:CB$, by equality of (*art. 53*) ratios, whence (*art. 52*) $FK=AB$. Rectangular parallelepipeds, which have their bases and altitudes reciprocally proportional, are then equal.

203. Since all parallelepipeds, right or oblique, which have equal bases and altitudes, are equal; it follows that all parallelepipeds, right or oblique, which have their bases and altitudes reciprocally proportional, are also equal.

204. Since a parallelepiped is equal to two triangular prisms, and a cylinder or prism of any base equal to as many triangular prisms as the base may be divided into triangles; it is evident, that all prisms or cylinders, which have their bases and altitudes reciprocally proportional, are equal.

205. Because a pyramid or cone is one third of the prism or cylinder of the same base and altitude; and

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whatever is true in respect to the wholes, (*art. 54*) is likewise so in respect to the parts, it follows, that pyramids or cones, which have their bases and altitudes reciprocally proportional, are also equal.

T H E O R E M.

206. *Similar parallelepipeds EL, FK, are as the cubes of their homologous sides EN, PF.*

For the cube of EN, and the solid EL, having the same altitude EN, are to each other as the square of EN to the rectangle NL; and the cube of PF and the solid FK, having the same altitude PF, are to each other as the square of PF to the rectangle PK: But by the similarity of the solids EL, FK, we have $EN:PF::NH:PI$, and $EN:PF::HL:IK$. Therefore (*art. 81*) the square of EN, is to the square of PF, as the rectangle NL of the remaining antecedents is to the rectangle PK of the remaining consequents; and since the solids are proportional to these planes, the cube of EN will be to the solid EL, as the cube of PF is to the solid FK. Similar parallelepipeds are then to each other as the cubes of their homologous sides.

207. Similar triangular prisms are to each other as the cubes of their homologous sides; since they are the halves of parallelepipeds, of the same altitudes, and a base double the base of the prism, and the parts are (*art. 54*) proportional to their wholes.

208. All similar prisms and cylinders are as the cubes of their axes or homologous sides; for cylinders are equal to (*art. 188*) prisms of equal bases and altitudes; prisms may always be divided into triangular ones, and the wholes are in the same ratio as their (*art. 54*) like parts.

209. All similar pyramids or cones are as the cubes of their axes, since they are one third of the prisms or cylinders of the same bases and altitudes, and the like parts are in the same ratio as their wholes.

210. All similar solids are in the triplicate ratio to that of their homologous sides; for they are as the cubes of

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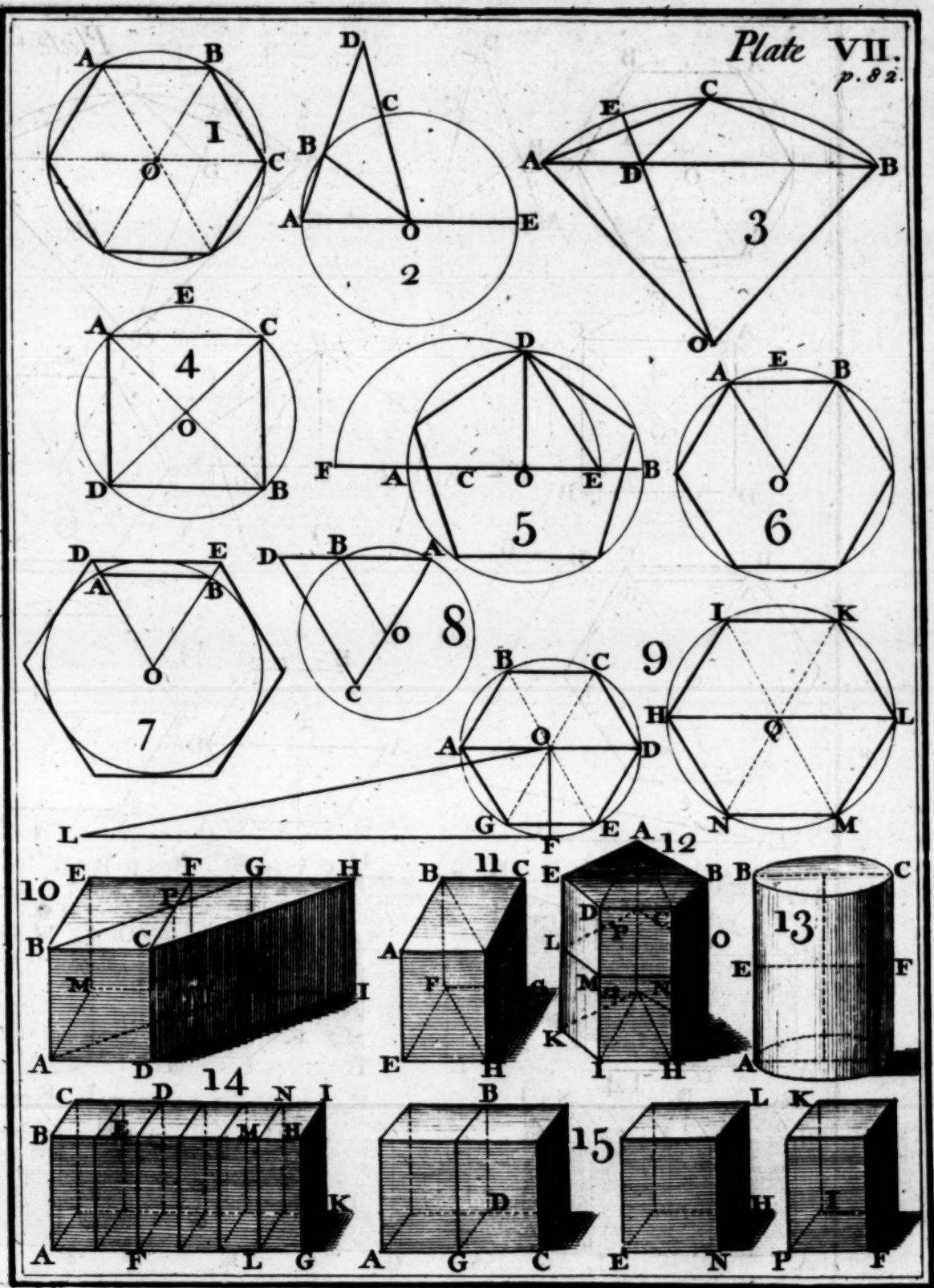
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THEOREM.

211. *The frustum ALKB, whose base AC is a square, is equal to a pyramid of the same base and altitude, and to another of the same altitude, whose base is a rectangle made by the sum AB, GH, of the sides of the opposite bases, and the side of the least.* Fig. 20.

It is evident, that this frustum is equal to a pyramid of the same base AC, and the same altitude EF, and to four pyramids such as ABHEG, whose base is the side AH, and vertex the point E.

The base AH is equal to half the rectangle made of the sum of the parallel sides AB, GH, (*art.* 51) and the perpendicular drawn in that plane to these parallel sides.

Let FESM be a section through the axis FE, perpendicular to the side AH; so that EF represents the axis, SM the perpendicular to the parallel sides, and ER perpendicular to MS produced, the altitude or perpendicular drawn from the vertex E to the base ABHG produced: If $AB + GH = A$, then will (*art.* 193) $\frac{1}{2} A \times SM \times ER$, express the pyramid AGEHA; and if SN, EP, are drawn parallel to EF and SM, the rectangle SF will be equal to the parallelogram EM, or $ES \times EF = ER \times SM$: Therefore $\frac{1}{2} A \times SM \times RE$, will be equal to $\frac{1}{2} A \times EF \times ES$, this will express one of these pyramids: And since ES is equal to the perpendicular drawn from the point E to GH, it will be equal to half the side GL or GH; if therefore, we substitute 4 ES, or 2 GH, for ES, we shall have $\frac{1}{2} A \times GH \times EF$, for the expression of the four pyramids, whose bases are in the sides of the frustum, and vertex in the point E. Consequently, the frustum is equal to a pyramid of the same base ABCD, and altitude FE; and to another, whose base is a rectangle made of the sum of the sides AB, GH, and the side GH and altitude EF that of the frustum.

212. Hence it is evident, that the square frustum AK , is equal to a pyramid whose base is $AB^2 + AB \times GH + GH^2$, and whose altitude is the same as that FE of the frustum: Whence the base is equal to the sum of the bases of the frustum, and to a mean proportional between these planes.

213. Since all frustums of cones or pyramids of any base, are equal to a square frustum, whose opposite bases and altitudes are equal, it follows that the frustum of any pyramid or cone is equal to a pyramid, whose base is equal to the sum of the opposite bases of the frustum and a plane which is a mean between these bases, and altitude equal to that of the frustum.

N. B. The mean plane in the frustum of a cone, is equal to half the rectangle made by the radius of one base and the circumference of the other; and in a frustum whose opposite bases are similar polygons, to half the rectangle made by the perpendicular drawn from the center in one, to one of the sides, and the sum of the sides in the other.

For if a, b , are the radii in the cone, or the perpendiculars in the pyramid, c, d , the circumferences or sums of their respective sides: Then will ac, bd , be double the upper and (*art.* 174) lower bases, and bc double the mean plane: For we have (*art.* 171) $a:b::c:d$, by supposition, and (*art.* 65) $ac:bc::a:b$, as likewise $cb:bd::c:d$; therefore, $ac:bc::bc:bd$, by (*art.* 53) equality of ratios.

T H E O R E M.

214. If about the quadrant AMB of a circle, a square $ADBO$ be described, and any line PN , be drawn parallel to the radius OB , meeting the diagonal OD in L , the circumference in M , and BD in N ; the segment of the sphere described by the segment $PMBO$ in the rotation of the figure about the axis AO , will be equal to the solid described by the space $LNBO$ in that rotation. Fig. 21.

As AD is equal to AO , PL will be equal to OP ; and by reason of the right angled triangle OPM , the dif-

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difference between the squares of OM and OP or PL, will be equal to the square of PM; or because OM is equal to PN by the property of the circle, the difference between the squares of PN, and PL, will be equal to the square of PM.

Now since circles are as the squares of their (*art.* 170) radii, the difference between the circles described by PN and PL in the rotation, or the area described by LN, will be equal to the circle described by PM; and as this property is general, the solid described by LNBO is equal to (*art.* 186) the solid described by PMBO.

215. Hence any segment PMBO of a sphere, is equal to the difference between the corresponding cylinder described by the rectangle PNBO and the cone described by the triangle OPL. Consequently, the semi-sphere described by the quadrant AMBO, is equal to the difference between the circumscribed cylinder described by the rectangle ADBO, and the inscribed cone, described by the triangle OAD.

216. Since the cone described by the triangle OAD is (*art.* 194) one third of the cylinder described by the rectangle ADBO, it is manifest, that the semi-sphere is two thirds of that cylinder. Consequently, the circumscribed cylinder, the semi-sphere, and the inscribed cone, are to each other as the numbers 3, 2, 1.

217. As the cones described by the triangles OPL, OPM, OPN, in the rotation of the figure about the axis AO, have the same altitude PO, they (*art.* 195) are as their bases, or as the squares of the radii (*art.* 170) PL, PM, PN; and since the difference between the squares of PN and PL, is equal (*art.* 170) to the square of PM; the solid described by the space OLN, will be equal to the cone described by OPM. If the solid described by OLN be taken from that described by OLNBO, and the cone described by OPM, from the segment of the sphere; the solid described by OBN, will be equal to the solid described by the sector OBM.

218. Hence, if these equal solids are supposed to be made up by an infinite number of small pyramids, whose

bases are in the surfaces described by BN, and BM, and vertexes in the center O: then the sum of all the pyramids in the one being equal to the sum of all those in the other; and their altitude OB the same; the sum of the bases in the one will be equal to the sum of the bases in the other. Consequently, the surface of any segment OBMP, of a sphere is equal to the surface of the corresponding part PNBO, of the circumscribed cylinder, and the surface of the whole sphere to the convex surface of the cylinder.

219. It is manifest, that spheres are similar, and therefore as the cubes of (*art.* 208) their axes or diameters; or in the triplicate (*art.* 210) proportion to that of their diameters; since the diameters of circles are as (*art.* 171) their circumferences.

T H E O R E M.

220. *The convex surface of a right cylinder is equal to a rectangle made of the circumference of the base and its axis.* Fig. 24.

For if we suppose this surface to unfold itself into a plane one, it is evident, that it would make a rectangle, whose base is equal to the circumference of the base of the cylinder and altitude to one of the sides or the axis; since all the sides are equal to each other and to the axis, by the definition of a right cylinder.

221. Hence, as the surface of the sphere is equal to the convex surface of the circumscribed cylinder, and the surface of that cylinder to the rectangle of the circumference of a great circle of the sphere and the diameter of the sphere; it follows, that the surface of a sphere is equal to a rectangle made of the circumference of a great circle and its diameter.

222. The area of a circle being (*art.* 174) the fourth part of a rectangle whose base is the circumference, and altitude the diameter, it is evident, that the surface of a sphere is quadruple the surface of its great circle.

223. Since the circles are as (*art.* 170) the squares of their radii, and the square of a line double another line,

is

is quadruple the square of that line; it is manifest, that the surface of a sphere is also equal to a circle, described with the diameter of the sphere as a radius.

224. Hence, the surfaces of similar cylinders, and of spheres, are to each other as the squares of their axes or diameters, or in the duplicate (*art. 77*) ratio to that of their axes: For the axes being proportional in the cylinders to the diameters of the bases, and the diameters of circles as their circumferences, they are as the squares (*art. 79*) of their homologous sides or axes.

T H E O R E M.

225. *The convex surface of a right cone is equal to a circular sector, whose radius is the side of the cone, and the arc equal to the circumference of the base of the cone.*

Fig. 17.

For if this surface be supposed to unfold itself into a plane one, it is evident, that, as all the sides BK of the cone are equal, it would form a circular sector, whose radius is equal to the side BK, and the arc to the circumference of the base of the cone.

226. Hence the surfaces of right similar cones are as the squares of their axes; for as they are equal to similar sectors, which are as the squares of the radii or sides BK, and these sides are as the axes AK, in similar cones.

S E C T. IX.

The INVESTIGATION of *AREAS* and *SOLIDS*.

MAGNITUDES are here considered as generated by motion; that is, a line, surface and solid, to be described by the motion of a point, line, and a surface respectively.

P R O B. I.

227. *If the area ANQ, terminated by the perpendicular NQ to the base AQ, be expressed by the base AQ and constant quantities, in such a manner, that when the base AQ becomes AP, the area ANQ becomes AMP; it is*

required to find the perpendicular PM from the given area AMP. Fig. 22.

Subtract the area ANQ from AMP, and divide the difference QNMP by the difference QP of the bases; then the quotient will give a line greater than QN and less than PM: For the rectangle of that quotient and QP, is equal to the area QNMP, and the rectangle PN is less, and QM greater than that area; and as they have the same base QP, the altitude QN will also be less, and PM greater than the said quotient.

Now as this always happens when QN becomes equal to PM, the quotient which is always greater than the one and less than the other, will likewise become equal to the perpendicular PM required.

P R O B. II.

228. *If the figure AM revolves about the axis AP, to find the plane or circle described by PM, in that rotation, from the given solid described by the area AMP.*

From the solid described by AMP, subtract the solid described by ANQ, and divide the difference described by QNMP, by the difference QP of the bases; then the quotient will, by the same argument as before, be a plane or circle greater than that described by QN, and less than that described by PM; and when AQ becomes AP, and QN equal to PM, this quotient will become equal to the circle required, described by PM.

It may be observed, that this problem not only extends to any solids described by the rotation of the figure AMP about the axis AP, but likewise to all solids whose sides of similar sections are always expressed by QN or PM; for the demonstration of this case is exactly the same with the former.

P R O B. III.

229. *If the figure AMP revolves about the line AT perpendicular to the base AP; to find the surface described by PM, from the given solid described by the area AMP.*

From the solid described by AMP, subtract the solid described by ANQ, and divide the difference described by

by ONMP, by the difference QP, of the bases; then by the same argument as in the first problem, the quotient will give a surface greater than that described by QN, and less than that described by PM; and when the bases AQ, AP, or the perpendiculars QN, and PM, become equal, the quotient will become the surface required, described by PM.

230. Hence if $AQ=z$, $AP=x$, and a, b, c, d , denote constant quantities; then if $az+bz^2+cz^3+dz^4$ expresses the area ANQ, or the solid described by that area about the axis AP, $ax+bx^2+cx^3+dx^4$, will express the area AMP, by condition, or the solid described by that area about the axis AP; and if the former be taken from the latter, the difference $ax-az+bz^2-cz^3+dx^4-dz^4$, will express the area QNMP, or the solid described by that area, which being divided by QP or $x-z$ gives,

$$x-z) ax-az (a.$$

$$x-z) bxx-bzz (bx+bz.$$

$$x-z) cx^3-cz^3 (cxx+cxz+cz^2.$$

$$x-z) dx^4-dz^4 (dx^3+dxz+dxz+dz^3.$$

Therefore $a+bx+bz+cxx+cxz+cz^2+dx^3+dxz+dxz+dz^3$ is the quotient; and when AQ becomes AP or $z=x$, this quotient gives $a+2bx+3cxx+4dx^3$ for the perpendicular or plane required.

231. Since the area or solid $ax+bx^2+cx^3+dx^4$, gives $a+2bx+3cxx+4dx^3$, for the perpendicular or plane; that is, ax gives a , bx^2 gives $2bx$, cx^3 gives $3cxx$ and dx^4 , gives $4dx^3$; it is manifest, that if the area or solid AMP, be expressed by any whole powers of the base AP and constant quantities, the perpendicular or plane PM, will be found by this

GENERAL RULE I.

232. Multiply every term by the exponent of the respective power of the base, and divide the whole by the base.

On the contrary, the perpendicular or plane PM, being expressed by any whole powers of the base AP, and

and constant quantities, the area or solid AMP, will be found by this.

GENERAL RULE II.

233. *Multiply every term by the base, and divide each term by the exponent of the respective power of the base thus increased.*

These general rules hold good, whether the exponents are whole or broken numbers, positive or negative. For when they are negative, divide by $-z+x$, and the quotients will be the same as before, only ne-

gative; and when they are fractions, make $x^{\frac{r}{n}}=u^r$, $z^{\frac{r}{n}}=y^r$; then $x=u^{\frac{n}{r}}$, $z=y^{\frac{n}{r}}$, and $x^{\frac{r}{n}}-z^{\frac{r}{n}}$ divided by $x-z$, becomes equal to $\frac{u^r-y^r}{u^n-y^n}$; the numerator di-

vided by $u-y$ gives ru^{r-1} when $x=z$ or $u=y$, by the first rule, and the denominator divided by $u-y$, gives nu^{n-1} , when $u=y$; hence ru^{r-1} divided by nu^{n-1} gives $\frac{r}{n}u^{r-n}$, or because $x^{\frac{r}{n}}=u^r$ and $x=u^{\frac{n}{r}}$, this

quotient becomes $\frac{r}{n}x^{\frac{r}{n}-1}$, which answers to rule the first.

234. *To find the area of a triangle KEF. Fig. 23.*

Let KD be perpendicular, and LN parallel to the base EF, the latter intersecting the former in Q; if the constant quantities $EF=b$, $KD=a$, and the variable $KQ=x$; we have (art. 67) $KD(a):KQ(x)::EF(b):LN=\frac{bx}{a}$. Now the perpendicular LN being

$\frac{bx}{a}$, we get $\frac{bx}{a}$, by the second rule, for the area of the triangle KLN, and $\frac{1}{2}ab$, when KQ becomes KD or $x=a$, for the area of the triangle KEF.

235. To

235. To find the content of a right cylinder, described by the rectangle ABCD about the side AB as an axis. Fig. 24.

From any point P in the axis, draw PM perpendicular to AB; call AP, x and the area described by the radius AD or PM, a : then the plane described by PM, being constant, we get $a x$ by the second rule for the cylinder described by AM, and $a \times AB$ for that described by the rectangle AC.

236. To find the content of the cone described by the right angled triangle ABC, about the side AB as an axis. Fig. 25.

From any point P in the axis, draw PM parallel to the base BC; make the constant quantities $AB = a$, $BC = b$, the circumference of the radius BC, c and $AP = x$. The similar triangles ABC, APM, give (art. 67)

$$AB (a) : AP (x) :: BC (b) : PM = \frac{bx}{a}; \text{ also (art. 171)}$$

$$b : \frac{bx}{a} :: c : \frac{cx}{a} = \text{to the circumference of the radius}$$

PM; hence we get (art. 174) $\frac{bx}{2a} \times \frac{cx}{a}$ or $\frac{bcxx}{2aa}$ for

the area or plane described by PM, and by the second rule, $\frac{bcx^3}{6aa}$ for the content of the cone described by APM,

and $\frac{1}{6} abc$ when AP becomes AB or $x = a$, for the cone described by ABC.

237. To find the surface of a right cylinder. Fig. 26.

From any point p in the base draw pn parallel to the axis; we are to find the surface described by the line pn from the given solid described by the rectangle An, according to the third problem.

If $AD = a$, $AB = b$, $Ap = x$, and c the circumference of the radius AD; then will (art. 171) $AD (a) : Ap (x) :: c :$

$\frac{cx}{a} =$ to the circumference of the radius Ap, and $\frac{cx}{2a}$ will (art. 174) be the area of the circle described by that

radius, and (art. 235) $\frac{bcxx}{2a}$ the content of the cylinder described by the rectangle An; now by the first rule

we

we get $\frac{bcx}{a}$ for the surface described by p n, and therefore bc for that described by the line DC, as in *art.* 220.

238. To find the surface of a right cone described by AC. Fig. 27.

Draw any where GH parallel, and BD perpendicular to AC, the latter intersecting the former in L; if BL be taken for the base, we are to find the surface described by GH, from the given solid described by the triangle GBH, about the axis AB according to the third problem.

If $BC=a$, $AC=b$, $BD=d$, $BL=x$, and c the circumference of the radius BC: the right angled similar triangles ABC, GLB (*art.* 69) give $BC(a):AC(b)::$

$BL(x):BG=\frac{bx}{a}$, and the parallels AC, GH (*art.*

67) $BD(d):BC(a)::BL(x):BH=\frac{ax}{d}$; but (*art.*

171) $BC(a):BH\left(\frac{ax}{d}\right)::c:\frac{cx}{d}=$ to the cir-

cumference of the radius BH; hence $\frac{ax}{2d} \times \frac{cx}{d}$ or $\frac{acxx}{2dd}$ expresses (*art.* 174) the area of the circle describ-

ed by BH, which gives $\frac{bx}{3a} \times \frac{acxx}{2dd}$ or $\frac{bcx^3}{6dd}$ for the cone (*art.* 236) described by the triangle GBH; and

by the first rule we get $\frac{bcxx}{2dd}$ for the surface described by GH; and when $x=d$, $\frac{1}{2}bc$ for that described by AC, as in *art.* 225.

Fig. 25. The circumferences being as their radii, that described by the point which bisects the side AC of the cone, will be half the circumference of the base, and therefore the surface of the cone is equal to the rectangle made by AC, and the circumference described by the point which bisects it; by the same reason, the sur-

surface of any part CM is equal to the rectangle made by that line, and the circumference described by the point which bisects CM.

239. To find the content of a segment of a sphere described by the segment of a circle OBMP, about the radius OA as an axis. Fig. 21.

Let the radius $OA = a$, its circumference c , $OP = x$, and $PM = y$; then $yy = aa - xx$, by the property of the circle (art. 112) and since (171) $OA (a) : PM (y) :: c : \frac{cy}{a} =$ to the circumference of the radius PM,

and $\frac{cyy}{2a}$ or its equal $\frac{c}{2a} \times aa - xx$ for the area of the circle (art. 174) described by PM; and by the second rule, $\frac{c}{2a} \times aa x - \frac{1}{3} x^3$ expresses the content of the segment required, which, when $x = a$, gives $\frac{1}{3} aac$ for the content of the semi sphere, as in art. 216.

240. Since $\frac{cyyx}{6a}$ or its equal $\frac{c}{2a} \times \frac{1}{3} aa x - \frac{1}{3} x^3$, expresses (art. 236) the cone described by the triangle OPM, which taken from $\frac{c}{2a} \times aa x - \frac{1}{3} x^3$ the segment described by OBMP, leaves $\frac{1}{3} acx$ for the solid described by the sector OBM.

241. To find the surface of a sphere. Fig. 28.

From the center O with any radius Oa, describe another quadrant am b, intersecting the radius OM in m, draw mp perpendicular to OA: then from the given solid described by the sector Obm, we are to find the surface described by mb, according to the third problem.

If Om = z, be the base, the arc mb the perpendicular, and the rest as before; the similar triangles OPM Op m, give OM (a) : Om (z) :: OP (x) : Op = $\frac{xz}{a}$,

and since (art. 171) $a : z :: c : \frac{cz}{a} =$ to the circumference

rence

rence of the radius Ob : by writing z , $\frac{cz}{a}$, and $\frac{xz}{a}$ for a , c , x , into $\frac{1}{3}acx$ found in the last article, we get $\frac{cxz^2}{3aa}$, for the solid described by the sector Omb , and

by the first rule, $\frac{cxzz}{aa}$ for the surface described by the arc mb , which gives cx for the surface described by the arc BM , as in *art.* 218.

242. To find the area of the circular segment $OBMP$.

Let the radius OA be unity, $OP=x$, then will $PM=\sqrt{1-xx}$, by the nature of the circle, and extracting the square root of $1-xx$, by the rule given in algebra, gives $PM=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}$

&c. and by the second rule $x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}$ — &c. for the area of the segment required.

243. Since the triangle OPM is equal to $\frac{1}{2}OP \times PM$, and $PM=1-\frac{x^2}{2}-\frac{x^4}{8}$ — &c. that triangle will be $=\frac{x}{2}-\frac{x^3}{4}-\frac{x^5}{16}-\frac{x^7}{32}-\frac{5x^9}{256}$ — &c. which taken from the

segment $OBMP$ leaves $\frac{x}{2}+\frac{x^3}{12}+\frac{3x^5}{80}+\frac{5x^7}{224}+\frac{35x^9}{2302}$, for the area of the sector OBM .

244. The sector OBM , is also equal to half the rectangle (*art.* 175) made by the radius OB (1) and the arc $BM=z$, that is $=\frac{1}{2}z$; these two values of the sector being made equal and multiplied by 2, gives $z=x+\frac{x^3}{6}+\frac{3x^5}{40}+\frac{5x^7}{112}+\frac{35x^9}{1152}$ + &c. or if $A=x$, $B=\frac{1}{2}Ax$, $C=\frac{3}{4}Bxx$, $D=\frac{5}{8}Cxx$, $E=\frac{7}{8}Dxx$, we get $z=A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\frac{1}{5}E$ +, &c.

If the arc BM be 30 degrees, then will $x=\frac{1}{2}=.5$, and $A=.5$, $B=\frac{1}{8}A$, $C=\frac{3}{16}B$, $D=\frac{5}{24}C$, $E=\frac{7}{32}D$, $F=$

$\frac{2}{40}$ E, $G = \frac{11}{48}$ F, $H = \frac{13}{30}$ G, $I = \frac{11}{24}$ G, these values reduced into decimals give

A = .5	.50000,000
B = .0625	2083,334
C = .01171,875	234,375
D = .00244,141	34,877
E = .00053,406	5,934
F = .00012,016	1,093
G = .00002,754	212
H = .00000,639	43
I = .00000,149	9
	.52359,877

This being the sixth part of half the circumference, which therefore multiplied by 6 gives 3.14159,262 for half the circumference of a circle whose radius is unity, true excepting the last figure 2 which should be 5. Hence, the radius is to half the circumference, or the diameter to the whole circumference, as unity to 3.14159, &c.

If the diameter be 7, the circumference will be 21.99 or 22 nearly, which is *Archimede's* proportion. If the diameter be 113, the circumference will be 354.9999, or 355 exceedingly near, which is that of *Snellius*.

A general method for finding the greatest or least value of an expression.

If an expression be such, that when the variable quantity increases, it also increases for a while and then decreases, it has a value greater than the rest; and if the expression decreases for a while, and then increases, whilst the variable quantity increases, it has a value less than any other. These values we are to find.

245. *To find the greatest or least value of an expression.*

Let the ordinate of a curve denote the expression, and the corresponding base, the variable quantity; then

then because the expression increases for a while, and decreases afterwards, or decreases for a while, and increases again by supposition, it is evident, that the curve has two equal ordinates, one before it passes the highest or lowest point, and another on the other side; now the values of these equal ordinates being subtracted from each other, their differences will be 0, and when divided by the difference between the two corresponding bases must also be 0. Consequently, the expression found by the first rule (*art. 232*) being made $= 0$, will determine the variable quantity answering to the greatest or least value of the expression.

246. Hence, if the value of the variable quantity, found by the first rule, is 0, the expression has no greatest nor least value; if the variable quantity has but one value, the expression has one greatest or one least value; and to know which it is, take any two values of the variable quantity, one greater and the other less than that found; then if the values of the expressions are both less than that found, it will be the greatest, but if they are both greater it will be the least.

And if the variable quantity has more than one value, the expression has greatest and least values alternately; but sometimes the least between two greatest is 0, and the greatest between two least, infinite.

247. *To divide a given line 2 a equally and unequally, such that the product of the unequal parts multiplied by the middle part, shall be the greatest possible.*

Let x be the middle part; then $a - x$, $a + x$, are the unequal parts, and $ax - x^3$, the product, which by rule the first, gives $ax - 3xx = 0$, or $aa = 3xx$. Because when x is 0 or $= a$, the expression $ax - x^3$ becomes 0; the value found is therefore the greatest.

248. Hence of all cones, that can be inscribed in a sphere, whose radius is a , that whose axis x , is such as $aa = 3xx$, will be the greatest. The same thing is true in regard to the greatest cylinder that can be inscribed in a whole sphere; since all inscribed cones or cylinders are in a constant ratio, to the above product.

249. To

249. To divide a line a into two parts, such that the square of one multiplied by the other, shall be the greatest possible.

Let x be one of these parts, then $axx - x^3$ must be the greatest possible; hence by the first rule $2ax = 3xx$, or $2a = 3x$.

Because when x is 0 or $=a$; the expression $axx - x^3$ becomes 0, which shews that the value found is the greatest. This holds true in any given number.

250. Hence, of all the cones that can be inscribed in a sphere, whose diameter is a , that whose axis x is the two thirds of the diameter a , is the greatest. For it is in a constant ratio to the product above.

251. Of all the triangles which have the same base and the same angle at the vertex, the isosceles contains the greatest area.

On the given base as a chord, describe a segment of a circle, so as to contain the given angle at the vertex; then of all the perpendiculars drawn from any point in the circumference to the chord; that which passes thro' the center and bisects it is the greatest.

252. Of all the rectilineal figures, whose sides are given, that which is inscribed in a circle is the greatest.

For if radii are drawn to the angles, it will be divided into as many isosceles triangles as it has sides, each of which being greater than any other triangle of the same base, and the same angle at the vertex, the sum of all or the whole figure will be the greatest.

253. Hence, it easily follows, that all figures whatsoever of the same boundary, the circle is the greatest. Also, of all equal areas, the circle has the least boundary. The same thing is true in regard to the sphere, that is, of all solids of the same boundary, the sphere is the greatest, and of all equal solids, the sphere has the least boundary. This method is of great consequence in mechanics; and in the perfections of machines, as will appear hereafter.

METHOD for computing the Number of Shot contained in a PILE.

Shot are generally piled in three different manners, the base is either a triangle, square, or a rectangle, and from thence the piles are called, triangular, square, and oblong.

If the triangular prism ABCD of shot (*fig. 30*) whose height BD or AC, is equal to the corner row AB, be cut diagonally by a plane CF; it is evident that the triangular pyramid CFD, is (*art. 193*) one third of the prism, and the remainder CFA, the two thirds.

Now because the number of shot in the triangular base ABC, is expressed by the terms of the arithmetical progression 1, 2, 3, 4, 5, &c. of the natural numbers whose last term is equal to z the number of shot in the base AC; and the sum of all the terms, to the sum $1+z$ of the first and last multiplied by $\frac{1}{2}z$ half their number; by *art. 34* of our algebra; that is, to $\frac{1}{2}zxz+1$.

The base ABC, multiplied by z the number of triangles contained in the prism ABCD, gives $\frac{1}{2}zxz+1$ for its content, and two thirds $\frac{2}{3}zxz+1$ of which, that of the part AFC. But it is evident, that the plane CF cuts the triangular range CB into two parts, the one CFB, is for the same reason as above, one third of $\frac{1}{2}zxz+1$ that range, which third $\frac{1}{6}zxz+1$, will express the part CFB; this being added to $\frac{1}{3}zxz+1$, found before gives $\frac{1}{3}zxz+1 + \frac{1}{6}zxz+1$, or $\frac{1}{2}zxz+1$ $\times 2z+1$, when reduced under the same denomination, for the square pile ABC or *fig. 29*. ABDC.

R U L E I.

Multiply the corner row, by the corner row more one, this product by twice the corner row more one, and divide by 6.

Example I. Let the side AC *fig. 29*, of a complete square pile be 36, then $36 \times 37 \times 73$ divided by 6, gives $6 \times 37 \times 73$ or 16206, for the required number of shot in that pile.

To find the content of the oblong pile ABDE. Fig. 20.

Let the base $AE = a$, and the breadth $= z$; then will $CE = a - z$, this length multiplied by $\frac{1}{2}xz + 1$, the triangular base EDL, gives $\frac{1}{2}z \times \frac{1}{2}xz + 1$, for the prism CBDE, to which add $\frac{1}{2}z \times \frac{1}{2}xz + 1 \times \frac{1}{2}z + 1$, the content of the square pyramid ABC, we get $\frac{1}{2}z \times \frac{1}{2}xz + 1 + \frac{1}{2}z \times \frac{1}{2}xz + 1 \times \frac{1}{2}z + 1$, or $3a - z + 1 \times \frac{1}{2}z \times \frac{1}{2}xz + 1$, when reduced under the same denomination, for the content required.

R U L E II.

From three times the length of the base more one, subtract the breadth, multiply the remainder by the product of the breadth, and the breadth more one, and divide by 6.

Example II. Let the length AE of a complete oblong pile be 50, and the breadth 20: then 3 times 50 more one, and 20 subtracted gives 131, and $131 \times 20 \times 21$, divided by 6 gives $131 \times 10 \times 7$ or 9170 for the number required.

To find the content of a triangular pile. Fig. 30.

Subtract the square pyramid ABC from the triangular prism ABCD; whose length BD is one more than the corner row AB; the remainder will be a triangular pyramid of the same altitude as the square one, and the base equal to the triangle ABC; whence the base $\frac{1}{2}xz + 1$, of the prism, multiplied by its altitude $z + 1$, gives $\frac{1}{2}z \times \frac{1}{2}xz + 1$, for its content; from which subtracting $\frac{1}{2}z \times \frac{1}{2}xz + 1 \times \frac{1}{2}z + 1$, the content of the square pyramid gives $\frac{1}{2}z \times \frac{1}{2}xz + 1 - \frac{1}{2}z \times \frac{1}{2}xz + 1 \times \frac{1}{2}z + 1$, or $\frac{1}{2}z \times \frac{1}{2}xz + 1 \times \frac{1}{2}z + 2$, when reduced, for that of the triangular pile.

R U L E III.

Multiply the base by the base more one, this product by the base more two, and divide by 6.

Example III. Let the base or corner row of a complete triangular pile be 20; then $20 \times 21 \times 22$ divided by 6, gives $10 \times 7 \times 22$ or 1540, for the number of shot required.

N. B. By dividing before the multiplication is performed, as we have done, and which is always possible the operation becomes shorter.

If the pile be incomplete, find its content as if it were so; then find the part wanting, and subtract it from the former. To find the sides of the part wanting subtract the corner row from the sides of the lower base and the differences will be the sides required.

Example I. Let the side of a square, unfinished pile be 25, and the corner row 18; then $25 \times 26 \times 51$ divided by 6, gives $25 \times 13 \times 17$ or 5525, by the first rule for the content of the compleat pile: now 18 taken from 25, leaves 7, and $7 \times 8 \times 15$ divided by 6, gives $7 \times 4 \times 5$ or 140 for the part wanting, which taken from 5525 leaves 5385 for the number required.

Example II. Let the length of an oblong unfinished pile be 25, its breadth 9, and corner row 6; then by the second rule, 3 times 25 more one and less 9, gives 67 and $67 \times 9 \times 10$ divided by 6, gives $67 \times 3 \times 5$, or 1005 for the compleat pile; now the corner row 6 taken from the side 25, and breadth 9, leaves 19 and 3; then 3 times 19 more, one less 3, gives 55 and, $55 \times 3 \times 4$ divided by 6, gives $55 \times 1 \times 2$, or 110 for the part wanting, which taken from 1005, leaves 895 for the required number.

Example III. Let the side of a triangular unfinished pile be 26, and the corner row 20; then $26 \times 27 \times 28$ divided by 6, gives $26 \times 9 \times 14$, or 13356 for the compleat pile: now 20 taken from 26, leaves 6, for the side of the part wanting, and $6 \times 7 \times 8$ divided by 6, gives 56, which taken from 13356, leaves 13300 for the number sought.

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BOOK II.

Of CONIC-SECTIONS.

SECT. I. PARABOLA.

DEFINITIONS.

HAVING placed a ruler EE , on a plane, and a square LD so as one of its sides DL touches the ruler, and a thread FMf , equal in length to the other side Df of the square being fastened with one end, to the end f of the ruler, and the other end to any point fix F in that plane on the same side of the ruler as the square. Plate IX. Fig. 1.

Then if the side DL slides along the ruler EE , whilst with a pin M , the part Mf of the thread is always kept tight and quite close to the edge of the square; the pin M will describe the part AM of a curviline.

If the square be placed on the other side of the point fix F , you may describe in the same manner, the part Am of a curviline; and the whole curve MAm is called a Parabola.

2. The point fix F is called the Focus.
3. The line AQ , drawn through the focus F perpendicular to the ruler EE , the Axis.
4. The line represented by the edge of the ruler EE , is called the Directrix.
5. A line double the distance BF , of the ruler from the focus, the Parameter.
6. And the part AP , of the axis, terminated by any perpendicular, Abscissa.
7. The point A where the curve meets the axis, the Vertex.
8. A line which bisects several parallel lines, terminated by any conic-section, is called a Diameter; and

the parallels, the Ordinates to that diameter; though their halves are commonly called so.

9. A right line which touches a conic-section but in one point, and when produced both ways. does not fall within the curve, a Tangent.

254. If from any of its points M , there be drawn two lines, the one to the focus F , and the other perpendicular to the directrix, they will be equal: for by taking from the length of the side fD of the square, and from the thread fMF its equal, the part fM , which is common to both, the remainders MD and MF are equal.

255. The distance FA between the focus and vertex, is equal to the distance AB , between the vertex and ruler. For suppose the square to fall on the axis AQ ; then because MF is always equal to MD , when MD becomes AB , its equal MF , becomes AF . Consequently, $AF = AB$.

THEOREM.

256. The square of any perpendicular MP to the axis, is equal to the rectangle made of the parameter and the corresponding abscissa AP . I say, $4AF \times AP = PM^2$.

For (art. 254) $FM = BP = AP + AF$; and $FP = AP - AF$; therefore $FM + FP = 2AP$, and $FM - FP = 2AF$. But because of the right angled triangle FPM , the difference between the squares of FM , FP , is equal (art. 49) to the square of PM , as likewise to the rectangle of the sum $2AP$ and difference $2AF$ (art. 46) of the sides FM and FP . Consequently, $2AP \times 2AF$, or $4AF \times AP = PM^2$.

If we call the parameter p ; then $4AF = p$, and $p \times AP = PM^2$.

257. Since the square of any perpendicular to the axis, is always equal to the rectangle made of the parameter p and the corresponding abscissa, it follows, that if NQ be drawn perpendicular to the axis, then will $p \times AQ = QN^2$. Consequently, $PM^2 : QN^2 :: (p \times AP : p \times AQ) :: AP : AQ$. Which shews, that the squares of the perpendiculars to the axis are always to each other as their corresponding abscissas.

258. The

258. The parabola opens continually as it recedes from the vertex A; for as the abscissa AP increases, so increases also the square of the corresponding perpendicular PM.

259. The axis bisects the parabola, for since the perpendiculars PM at equal distances AP, from the vertex A, are always equal on both sides the axis, the part of the parabola on one side of the axis is equal to the other. Consequently, the axis is a *diameter* which bisects its ordinates at right angles.

P R O B.

260. To draw a tangent to any point M of the parabola. Fig. 2.

Draw ML parallel to PA, and equal to the distance MF from the point M to the focus F, join FL; then the line MD, which is at right angles to the line FL will be the tangent required. Let LE be perpendicular to AP; then (*art. 254*) $ML = EP = MF$ by construction: and if from any other point m in DM, a line ml be drawn parallel to ML, and another to the focus F, it is evident that the line m L is equal (*art. 15*) to m F: now because of the right angled triangle m l L, the hypotheneuse m L is always greater than the side ml; but this line should be (*art. 254*) equal to m F, if the point m were in the curve; therefore the line MD touches the parabola only in the given point M.

261. If the tangent produced meet the axis in T, then because the angles FMT, LMT, are (*art. 16*) equal, as likewise the alternate ones FTM, TML, the angles FMT, FTM, are equal; and hence the sides FM and FT (*art. 26*) are equal; but (*art. 254*) $FM = EP$, and therefore $FT = EP$, and because (*art. 255*) $AF = AE$, we have $AT = AP$. Consequently, the subtangent PT is double the corresponding abscissa AP.

262. If MK be perpendicular to the tangent at M, then because of the parallels MK, LF, we have $ML = FK =$ (*art. 254*) EP ; and taking away the common line FP, we shall have $PK = EF$; that is, the part PK,

terminated by MP and the perpendicular MK to the tangent, is equal to half (art. 256) the parameter p.

263. The right angled similar triangles TPM, TMK, give TP or $2AP : TM :: TM : TK = 2FK = 2FM$; and therefore $AP : TM :: TM : 4FM$. Since the rectangle of the means, and that of the extremes, are the same in both these proportions,

L F M M A.

264. If from any point M in the parabola, there be drawn a tangent MT, an ordinate MP to the axis AQ, and a line MR, parallel to the axis; and if from any other point L, an ordinate LQ be drawn, meeting MR in R, and a line LK parallel to the tangent MT, meeting MR in E, and the axis in V; then will the triangle ERL be always equal to the corresponding parallelogram TMEV. Fig. 3.

The similar triangles TPM, VQL, are as the squares of their homologous (art. 88) sides PM, QL; and these squares are as (art. 257.) the corresponding abscissas, viz. $\frac{1}{2}$ TP, or (art. 261) $AP : AQ :: TMP : VQL$, and the triangle TMP, is to the space TMRQ, as $AP : AQ :: TMP : TMRQ$; therefore $VQL = TMRQ$ by equality of (art. 53) ratios; taking away the common space VERQ, we shall have $TMEV = ERL$.

T H E O R E M.

265. The square of any line EL parallel to the tangent TM, is to the square of that tangent, as the part ME of the line MR is to AP, or AT.

For the similar triangles ERL, TPM, are as the squares of their homologous sides (art. 88) $EL^2 : TM^2 :: ELR : TMP$; and the parallelogram TMEV is to the triangle TMP, as ME is to $\frac{1}{2}$ AT or AP; that is, $TMEV : TMP :: ME : AP$; but $TMEV = ELR$ by art. 264. Therefore $EL^2 : TM^2 :: ME : AP$, by equality of ratios.

266. If from the point M, a line be drawn to the focus F, then because (art. 263) $AP : TM :: TM : 4FM$, or $TM^2 = 4AP \times FM$; this value being substituted into the last proportion inverted, gives $AP : ME :: 4AP \times FM : EL^2$, or $EL^2 = 4FM \times ME$.

267. If

267. If the line LE produced, meets the parabola in K, it is evident that the square of EK is also equal to the rectangle $4 FM \times ME$; and therefore $EL^2 = EK^2$, or $EL = EK$; for whether the triangle ELR be placed on one side or the other of the line MR, the demonstration in art. 265 will be the same. Consequently, the line MR is a diameter, and the lines LK, parallel to the tangent at M, are its ordinates: *Which shews that the diameters are parallel to the axis in the parabola.*

268. If we call $4 FM$, p ; then because $p \times ME = EL^2$; it is evident, that whether ME be a diameter or the axis, the square of any ordinate EL is always equal to the rectangle made of the corresponding abscissa ME and the parameter p ; and the parameter of any diameter or axis is always equal to four times the distance MF of the vertex of the diameter or axis from the focus F.

269. If another ordinate NB be drawn to the diameter MR, then will $p \times MB = NB^2$. Which shews that the squares of the ordinates, to any diameter or axis, are to each other as the corresponding abscissas.

270. If from any points N, L, in the parabola, lines NC, LD, are drawn parallel to the diameter, meeting the tangent in C, D, the squares of the parts MC, MD, of the tangent terminated by these lines will be to each other as the lines CN, DL; for $MC = 3N$, $MD = EL$, and $CN = MB$, $DL = ME$.

T H E O R E M.

271. If a line Ll intersects a diameter MR, either within or without a parabola, the rectangle of its parts terminated by that diameter, will be equal to the rectangle made of the part MR of the diameter terminated by the line, and the parameter p , of the diameter AQ, to which that line is an ordinate, viz. $p \times MR = LR \times RL$. Fig. 4.

From the vertex M, of the diameter, draw the ordinate MP to the diameter AQ; then (art. 268) $p \times AP = PM^2$, and $p \times AQ = QL^2$; and therefore $p \times AQ - p \times AP$, or $p \times MR = QL^2 - PM^2$; but the difference between

between two squares is equal to the rectangle of the (art. 46) sum and difference of their sides; and because (art. 267) $QL = QI$, there will be $RI = QL + PM$, and $RL = QL - PM$. Consequently, $p \times MR = LR \times RI$.

272. Hence, if another line Kk be drawn parallel to Ll , intersecting the diameter MR in S , then will (art. 271) $p \times MS = KS \times Sk$: Consequently, $p \times MS : p \times MR$, or $MS : MR :: KS \times Sk : LR \times RI$. Which shews that if any diameter intersects two parallel lines, either within or without the parabola, the rectangles of the parts of these lines, terminated by the diameter, are as the parts of that diameter, terminated by these lines.

273. If any line $m n$ intersects the diameter MR in the same point S as the line Kk , and q be the parameter of the diameter to which $m n$ is an ordinate, then will $q \times MS = mS \times Sn$. Therefore $q \times MS : p \times MS$, or $q : p :: mS \times Sn : KS \times Sk$.

274. If the line $m n$, produced if necessary, meets Ll in D ; then will ($q : p ::$) $mS \times Sn : KS \times Sk :: nD \times Dm : LD \times DI$. In general, the rectangles of the parts of any two parallel lines terminated by the parabola, and any other line, will always be as the rectangles of the parts of that line terminated by the parabola and the parallel lines, whether they meet or intersect each other within or without the parabola.

P R O B.

275. To find the area of a parabolic space AMP , terminated by any ordinate PM to the axis AP . Fig. 5.

Draw AQ perpendicular, and MQ parallel to the axis; if $AP = x$, $PM = y$, and the parameter unity; then (art. 256) will $x = yy$: Now since $AQ = PM = y$, and $QM = x = yy$, the area AMQ , whose perpendicular QM is expressed by the square yy , of the base AQ , will be (art. 233) $\frac{1}{3} y^3$, or because $x = yy = \frac{1}{3} xy$; that is, the external space AMQ is one third of the rectangle PQ ; therefore the internal space AMP is the other two thirds of that rectangle.

276. If

276. If c be half the circumference whose radius is unity, then because the areas of circles are as the squares of the radii, we shall have cyy for the area described by their radius PM ; or because $x=yy$, that area will be expressed by cx ; which multiplied by x , and divided by 2 the exponent of x , gives $\frac{1}{2}cx^2$, or because $x=yy$, $\frac{1}{2}cxyy$ will express the content of the solid described by the semi-parabola AMP about the axis AP : which therefore is half the cylinder described by the rectangle PQ in that rotation, consequently the solid described by the outward space AMQ the other half, which is equal to the former.

277. Since the area of the circle described by QM (x) about the tangent AQ is cxx , or cyy^2 , because $x=yy$, which multiplied by y , and divided by its exponent 5, gives $\frac{1}{5}cyy^3$ or $\frac{1}{5}cxyy^2$ for the solid described by the space AMQ about AQ ; which is the fifth part of the cylinder $cxyy$ described by the rectangle PQ about that axis.

SECTION II. ELLIPSIS.

DEFINITIONS.

8. **H**AVING fastened the two ends of a thread FMf , to two points, fix F, f , in a plane, in such a manner as the interval Ff , between these points, be less than the length of the thread; then if with a pin M the thread is always kept tight, by moving this pin round the points fix, it will describe a curve line $MANa$, which is called an Ellipsis. Fig. 6.

2. The points fix F, f , are called the Foci.

3. The line Aa , which passes through the foci F, f , and is terminated by the ellipsis, is called the first or great Axis.

4. The point C , which bisects the axis, is called the Center.

5. The line Bb , drawn through the center C at right angles to the first axis Aa , and terminated by the ellipsis, is called the second or little Axis.

6. The

6. The line which is a third continued proportion to half the first and second axes, is called the Parameter to the first axis.

7. A diameter, which is parallel to the ordinates of another diameter, is called its Conjugate.

278. It follows, from the first definition, that if from any point M in the ellipse, there be drawn two lines to the foci F, f , their sum will always be the same, let that point be taken where it will.

279. When the point M falls at A , the part FM of the thread will become $=AF$, and the other part $fM = fA$; so that $fA + FA$ is equal to the length of the thread; and when the point M falls on a , the part fM becomes $=fa$, and the part $FM = Fa$; therefore $fA + FA = Fa + fa$; and by taking away the common line Ff , we have $2AF = 2af$, or $AF = af$.

280. Hence, the first axis Aa is equal to the thread fMF ; for because $fA + FA$ is equal to it, and $FA = fa$; we have $fA + FA = fA + fa = Aa$.

281. The distances of the foci, f, F , from the center C are equal; for because $CA = Ca$ by defin. 4. and $AF = af$; CF will be $= Cf$.

282. The lines drawn from either of the foci f , to the extremities of the second axis B, b , are each equal to half CA the first axis: for it is clear that when the point M falls on B or b , the two right angled triangles FCB, fCB , having the sides fC, CF , equal, and CB common to both, have their (*art. 15*) hypotenuses FB, fB , equal; that is, FM becomes equal to fM , and their sum being equal to the first axis, their half FB , or fB , will be $=CA$. Hence also,

I. The second axis Bb is bisected by the center C ; for since $fB = fb$, and Cf common to both right angled triangles BCf, bCf , they (*art. 15*) are equal in all respects. Therefore $BC = Cb$.

II. The second axis Bb is always less than the first Aa ; for since $FB = CA$, the hypotenuse FB will be always greater than any one of the sides BC . There-

fore

fore $2BC$ or Bb is less than $2CA$ or Aa ; and when the axes become equal, the ellipsis becomes a circle.

III. The square of half the greatest axis, is equal to the sums of the squares of half the least CB , and that of the distance CE , of the focus from the center.

IV. The difference between the squares of FB half the first axis and the distance CF , of the focus from the center, is equal to the square of half CB the second axis; or because $Fa = FB + FC$, and $FA = FB - FC$, we have (art. 46) $AF \times Fa = BC^2$.

T H E O R E M.

283. If from any point M a perpendicular MP be drawn to the first axis Aa , and CX be taken a fourth proportional to CA , CF , and CP , then will $AX = FM$, and $aX = Mf$.

Because in the triangle FMf , the rectangle of the sum and difference of the sides FM , Mf , is (art. 46) equal to the rectangle of the sum and difference of the segments FP , Pf ; but (art. 280) $2CA$ is the sum of the sides FM , fM , if D be half their difference, then as $2CF = fF$, and $2CP = fP - PF$. Therefore $2CA \times 2D = 2CF \times 2CP$; or $CA : CF :: CP : D = CX$ by supposition.

Therefore, since AC is (art. 280) half the sum of the lines FM and fM , and CX half the difference, AX will be equal to the least FM , and aX equal to the greatest fM .

T H E O R E M.

284. The square of the perpendicular MP , drawn from any point in the ellipsis to the first axis, is to the rectangle of the corresponding parts of that axis, as the square of half the second axis is to the square of half the first. I say, $PM^2 : APa :: CB^2 : CA^2$.

Because (art. 283) CX or (art. 280) $CA - FM : CP :: CF : CA$, by adding the antecedents to their consequents, we get $aP - FM : aF :: CP : CA$, and subtracting the antecedents from their consequents I. $FM + FP : AP :: aF : CA$.

Now

Now if in the first proportion the antecedents are subtracted from their consequents, there will be $FM = AP:CP::AF:CA$; and by adding antecedents to antecedents, and consequents to consequents, II. $FM = FP:aP::AF:CA$.

If the terms of the proportion I. are multiplied by the respective terms of the proportion II. and we take the square of PM , instead of the rectangle (*art. 46*) made of the sum and difference of the sides FM and FP of the right angled triangle FPM , as likewise the square of CB instead of the (*art. 282*) rectangle $AF \times Fa$; we shall have $PM^2 : APa :: CB^2 : CA^2$.

285. Since the square of any perpendicular PM is always to the rectangle of the corresponding parts AP, Pa , of the axis, as the square of CB to the square of CA ; when the perpendicular passes through the focus F , as LF , then the above proportion becomes $LF^2 : AFa :: CB^2 : CA^2$; or because AFa (*art. 282*) is $= CB^2$; we have $LF^2 : CB^2 :: CB^2 : CA^2$, or (*art. 82*) $LF : CB :: CB : CA$; therefore LF is the parameter of CA .

286. Because (*art. 176*) $AC^2 : CB^2 :: CA : LF$; we have $PM^2 : APa :: LF : CA$, (*art. 282*) by equality of ratios. Which shews, that the square of any perpendicular to the first axis is to the rectangle of the corresponding parts of the axis as the parameter LF is to half the first axis.

287. If there be drawn another perpendicular NQ to the first axis, then will $QN^2 : AQa : LF : CA$; and as $PM^2 : APa :: LF : CA$, it is evident that $PM^2 : APa :: QN^2 : AQa$, by equality; or $PM^2 : QN^2 :: APa : AQa$, by alteration. Consequently, the squares of the perpendiculars to the first axis, are as the rectangles of the corresponding parts of the axis.

288. The perpendiculars on either side the first axis, which are equidistant from the center, are equal; for if $CP = CQ$, then will $AP = aQ$, and $AQ = aP$; therefore $APa = AQa$, and $PM^2 = QN^2$, or $PM = QN$. Consequently, the axis is also a diameter, and the perpendiculars PM, QN , to it, its ordinates.

289. Fig. 8. If from any point M in the ellipsis, a line

line MR be drawn perpendicular to the second axis Bb; then because $CR = PM$, and $RM = CP$; and since the rectangle APa is equal to the difference (*art.* 46) $CA^2 - CP^2$ between the squares of half CA the axis, and the part CP; therefore $PM^2 : CA^2 - CP^2 :: CB^2 : CA^2$; or by substituting for CP, PM, their equals RM, CR, we shall have $CR^2 : CA^2 - RM^2 :: CB^2 : CA^2$; and comparing the difference between the consequents to that of the antecedents, to the terms of the last ratio, we shall have $RM^2 : BRb :: CA^2 : CB^2$; because $CB^2 - CR^2 = BRb$. Which shews, that the second axis is the conjugate diameter to the first axis, and that the perpendiculars MR to that axis, are its ordinates.

290. Since the square of any ordinate MR to the second axis, is to the rectangle of the corresponding parts of that axis, as the square of half the first is to the square of half the second, the properties of this axis, in respect to its ordinates, are the same as those of the first. Hence,

I. The lines drawn through the extremities of either axes, parallel to the other, touch the ellipsis in those points, for because the ordinates on both sides of the same axis are constantly equal, the lines parallel to them fall entirely without the ellipsis.

II. Each axis divides the ellipsis into two equal parts, so that if the one half was laid over the other, it would agree with it in every point.

P R O B.

291. To draw a line MD which shall touch the ellipsis in any given point M. Fig. 7.

From the foci f, F, draw lines to the given point M; in fM produced take $ML = MF$; then the line MD perpendicular to FL will be the tangent required.

From any other point m in MD, draw lines to the foci and to the point L; then because $Lm + mf$, or its equal $Fm + mf$, is always greater than (*art.* 23) fL or (*art.* 280) Aa ; and as the sum of these lines drawn from any point in the ellipsis to the foci, is (*art.* 280) always equal to Aa , it follows, that the point m

is

is not in the ellipsis; and consequently MD touches the ellipsis in the only point M.

292. If the tangent produced both ways meets the axes in T, t, it is manifest, that the angles fMt , FMT , made by the lines drawn from the foci to the point of contact M and the tangent, are equal.

For $ML=MF$ by construction, and MD perpendicular to FL, the angles LMD, FMD, are (art. 18) equal; and the angle LMD is equal to its opposite one fMt . Consequently, the angles FMD, fMt , are equal.

THEOREM.

293. If from the point of contact M, an ordinate MP be drawn to the first axis, half this axis will be a mean proportional, between the distances, of this ordinate, and the point T where the tangent meets the axis, from the center. I say, $\therefore CP:CA:CT$.

Draw CD; then because $CF=Cf$, and (art. 291) $FD=DL$, the line CD will be parallel to fL and equal to half of it, or (art. 280) $CA=CD$; and because of the parallels fM , CD, we have $fM:CD$, or $CA::fT:CT$; or $fM-CA:fC::CA:CT$. And if AX be taken equal to FM, then will (art. 283) $fM=aX$, or $fM-CA=CX$; but because (art. 283) $CX:CF::CP:CA$, we have $CP:CA::CA:CT$, be equality of ratios: since $CA=Ca$, and $Cf=CF$.

294. If the line Kk, drawn perpendicular to the tangent at M, meets the first axis in K and the second in k; then because of the parallels LF, MK, and fM , CD, we have CD or $CA:CF::fM:fK$; or subtracting the terms of the last ratio from those of the first, $CA:CF::fM-CA:CK$; but (art. 283) $CA:CF::CP:CX$: therefore, multiplying the terms of the first ratio by the respective terms of the second, then will $CA^2:CF^2::CP:CK$; because $fM-CA=(art. 283) CX$.

295. Since $CA^2:CF^2::CP:CK$, we get by subtraction $CA^2:CA^2-CF^2::CP:PK$; or because (art. 282) $CA^2-CF^2=CB^2$; we shall have $CA^2:CB^2::CP:$

CP:

CP:PK: and by the similarity of triangles CP or PM:PK::pk:PM. Therefore $CA^2:CB^2::pk:PM$, by equality of ratios; or $CB^2:CA^2::Cp:pk$.

296. Because (art. 65) $CP \times CT:PK \times CT::CP:PK$, we have $CA^2:CB^2::CP \times CT:PK \times CT$ by equality of ratios: but since (art. 293) $CP \times CT = CA^2$; we have also $CB^2 = PK \times CT$; or, $PK:CB::CB:CT$. The similar triangles MPK, TCt, give $PM:PK::CT:Ct$. Therefore (art. 85) PM or $Cp:CB::CB:Ct$. Which shews that the properties of the tangents are the same in respect to both axes.

T H E O R E M.

297. Any line MN passing through the center C, and being terminated by the ellipsis, will be bisected by the center. I say, $CM=CN$. Fig. 8.

By drawing MP, NQ perpendicular to the axis Aa, the right angled triangles CPM, CQN, will be similar, whence if $CP=CQ$, then will $PM=QN$, and $CM=CN$; but since the ordinates which are equidistant from the center are equal, it is evident that if the point M is in the ellipsis, the point N is also in the ellipsis. Consequently MN is bisected by the center.

L E M M A.

298. If from the extremity L of any ordinate LQ to the first axis a line LV be drawn parallel to the tangent MT, and from the point M of contact, a line Mm through the center, intersecting LV in E, and the ordinate LQ, produced if necessary, in R; the quadrilateral TMEV will be always equal to the corresponding triangle LER. I say, $TMEV=LER$. Plate X. Fig. 9.

The similar triangles TPM, VQL, are as the squares (art. 88) of their homologous sides PM, QL, or (art. 287) $TPM:VQL::CA^2-CP^2:CA^2-CQ^2$; and the triangles CPM, CMT, having the same altitude PM, are as their bases CP, CT, or because $\therefore CP:CA:CT$: as the squares of CP and CA. Since therefore the triangles, CQR, CPM, and CMT, are as the squares of CQ, CP, CA; we have by subtraction

traction $TPM : QRMT :: CA^2 - CP^2 : CA^2 - CQ^2 ::$
 $TPM : VQL :$ whence $VQL = QRMT$; and by sub-
 tracting the common space $VERQ$, there will remain
 $TMEV = LER$.

299. Since this property is always the same, wherever
 the line LV is drawn parallel to the same tangent; if
 it passes through the center, or becomes Nn , the space
 $TMEV$ becomes TMC , and the triangle LER the tri-
 angle NCr . Therefore $TMC = CNr$.

T H E O R E M.

300. *The square of any line LE , drawn parallel to a
 tangent MT , terminated by the line Mm which passes
 through the center and the point of contact, will be to the
 rectangle of the corresponding parts of the line Mm , as the
 square of CN , parallel to the tangent is to the square of
 CM . I say, $LE^2 : MEm : CN^2 : CM^2$.*

The similar triangles CEV , CMT , give (*art. 69*)
 $CM^2 : CE^2 :: CMT : CEV$; subtracting the consequents
 from their antecedents, and comparing the antecedents
 to the differences $CM^2 : CM^2 - CE^2 :: CMT : TMEV$.
 But the similar triangles CNr , LER , give (*art. 69*)
 $CN^2 : LE^2 :: CNr : LER$; and because (*art. 299*)
 $CMT = CNr$, and (*art. 298*) $TMEV = LER$, we have
 $LE^2 : CN^2 :: CM^2 - CE^2 : CM^2$, by equality of ratios;
 because the line Mm is bisected by the center C , the
 rectangle MEm is equal to (*art. 46*) $CM^2 - CE^2$. Con-
 sequently, $LE^2 : MEm :: CN^2 : CM^2$, by altereration.

301. If from any point L in the ellipsis, a line is
 drawn parallel to the tangent MT , the square of that
 line LE is always to the rectangle MEm of the cor-
 responding parts of Mm ; in the given ratio, of the
 squares of CN and CM , of which side soever the point
 L is taken in respect to the line Mm ; hence if LE is
 produced, so as to meet the ellipsis in another point K ,
 the line LK will be bisected by Mm : for the square of
 KE will be to the rectangle MEm as CN^2 to CM^2 ; and
 the square of LE is to the same rectangle MEm , as
 CN^2 to CM^2 ; it follows that $KE^2 = LE^2$, or $KE = LE$.
 Consequently, Mm is a diameter, and Nn its conjugate.

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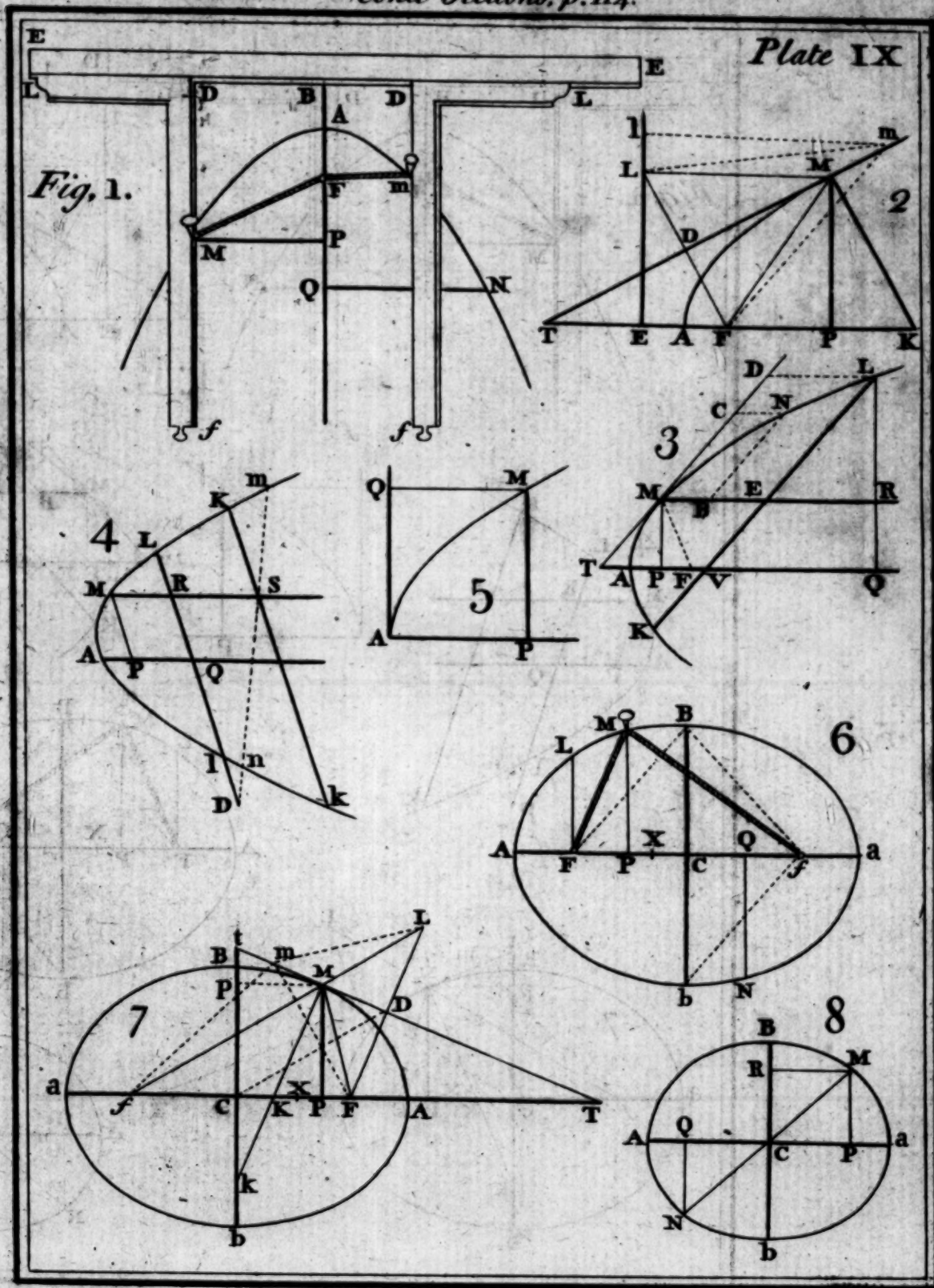
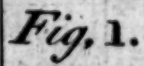
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Which shews that all diameters in the ellipsis pass through the center.

302. Because the property of any conjugate diameters Mm, Nn, is the same as that of the axes: it follows.

I. That all the ordinates, equidistant from the center, on either side of the diameters, are equal.

II. That the squares of any two ordinates are to each other as the rectangles of the corresponding parts.

III. Lastly, all what has been proved in respect to one diameter is likewise true in respect to its conjugate.

T H E O R E M.

303. *If a line Ll intersects any diameter Mm, either within or without the ellipsis, the rectangle of its parts, terminated by the diameter, will be to the rectangle of the corresponding parts of the diameter, as the square of the semi-diameter CN, parallel to this line, is to the square of half the diameter which it cuts. I say, $LRl : MRm :: CN^2 : CM^2$. Fig. 10.*

Let Aa be the diameter to which Ll is an ordinate, draw Nu and RS parallel to the tangent MT, the first intersecting the diameter Mm in F, the rest being the same as before; then the similar triangles QRS, QLV, and CNu, give $QL^2 : QR^2 :: QLV : QRS$; subtracting the consequents from the antecedents, and comparing the antecedents to the respective differences $QL^2 : QL^2 - QR^2 :: QLV : RLVS$, and $QL^2 : CN^2 : QLV : CNu$; therefore I. $CN^2 : QL^2 - QR^2 :: CNu : RLVS$, by equality of ratios.

The similar triangles CMT, CRS, give $CM^2 : CR^2 :: CMT : CRS$; subtracting the consequents from the antecedents, and comparing the antecedents to the respective differences II. $CM^2 : CM^2 - CR^2 :: CMT : RMTS$.

But (art. 298) $TMEV = LER$; by adding VERS to both, we get $RMTS = RLVS$; and since (art. 298) $TMFu = CNF$; by adding CFu to both, then will $CMT = CNu$. Therefore, the two former proportions I, II, give $CN^2 : QL^2 - QR^2 :: CM^2 : CM^2 - CR^2$;

CR^2 ; or because (*art.* 46) $QL^2 - QR^2 = LRl$, and $CM^2 - CR^2 = MRm$; it will alternately be $LRl : MRm :: CN^2 : CM^2$.

304. Fig. 11. If through the same point R the line Kk be drawn parallel to the diameter CX, then will $CM^2 : MRm :: CN^2 : LRl$, and $CM^2 : MRm : CX^2 : KRk$. Therefore by equality of ratios. $CN^2 : LRl :: CX^2 : KRk$.

305. If the line Hh, parallel to Ll, intersects the diameter Mm, in r, either within or without the ellipsis, then will $MRm : Mrm :: LRl : Hr h (:: CM^2 : CN^2)$.

306. If the lines Kk, Hh, produced if necessary, meet in D, either within or without the ellipsis, then will $KDk : HDh :: KRk : LRl$: in general, the rectangles of the parts of two parallel lines, are always to each other as the rectangles of the respective parts of any other line which intersect these parallels; and if one or two become tangents, these rectangles become squares: This is a natural consequence, from what has been said above.

T H E O R E M.

307. If from the point of contact M, an ordinate MP be drawn to the axis Aa, the rectangle of the parts CT, CP, terminated by the tangent, the ordinate, and the center, will be equal to the rectangle of the corresponding parts of the axis terminated by the ordinate. I say, $CP \times PT = APa$. Fig. 12.

Because (*art.* 293) $CT : CA :: CA : CP$, we have by subtraction $TA : CA :: AP : CP$, by adding antecedents to antecedents, and comparing the respective sums to the terms of the second ratio, we have $PT : AP :: Pa : CP$; or, $CP \times PT = APa$.

308. If from the extremity N of the diameter parallel to the tangent, a tangent Nt, and an ordinate NQ be drawn to the same axis Aa, the similar triangles TMC, tNC, give $CT : Ct :: PM : QN$; and because $\div CP : CA : CT$, as likewise $\div CQ : CA : Ct$, we have (*art.* 86) $CT : Ct :: CQ : CP$. Therefore,

CQ;

$CQ:CP::PM:QN$, by equality of ratios; or $CQ \times QN = CP \times PM$.

309. Hence, the square of CQ is equal to the rectangle of APa : for the similar triangles TPM , CQN , give (*art.* 69) $TP:CQ::PM:QN$; and since $CQ:CP::PM:QN$, we have $TP:CQ::CQ:CP$; (*art.* 307) but $TP:AP::Pa:CP$; therefore (*art.* 86) $AP:CQ::CQ:CP$; or, $APa = CQ^2$. It may be proved in the same manner that $AQ:CP::CP:Qa$; or $AQa = CP^2$.

310. Since (*art.* 284) APa , or (*art.* 309) CQ^2 : $PM^2::CA^2:CB^2$, we have (*art.* 82) $CQ:PM::CA:CB$; and by the same argument $CP:QN::CA:CB$.

T H E O R E M.

311. *The parallelogram GH , made by any two conjugate diameters Mm , Nn , is always equal to the rectangle made by the two axes Aa , Bb .*

Let the tangent HM meet the axis Aa in T , draw the ordinates MP , NQ , to that axis, and from the center C draw Cr perpendicular to the tangent TH ; then the similar triangles TCr , CQN , give $CN:QN::CT:Cr$; but (*art.* 293) $CP:CA::CA:CT$, and (*art.* 310) $CP:CA::QN:CB$; therefore $QN:CB::CA:CT$, by equality of ratios; and the first and last proportion give (*art.* 86) $CN:CB::CA:Cr$; or (*art.* 72) $CN \times Cr = CA \times CB$. But since $CN \times Cr$ is equal to the fourth part $CMHN$, of the parallelogram GH , and $CA \times CB$, the fourth part of the rectangle made by the axes; it follows, that the rectangle made by the two axes, is equal to the parallelogram made by two conjugate diameters.

T H E O R E M.

312. *The sum of the squares of any two semi-conjugate diameters CM , CN , will be always equal to the sum of the squares of the two semi-axes CA , CB .* Because of the right angled triangles CPM , CQN : we have $CP^2 + PM^2 = CM^2$, and $CQ^2 + QN^2 = CN^2$. Therefore $CM^2 + CN^2 = CP^2 + CQ^2 + PM^2 + QN^2$. But (*art.* 309) $CQ^2 = APa$, (*art.* 46) $CA^2 = CP^2$, or $CA^2 =$

$CQ^2 + CP^2$; and by the (*art.* 309) same argument $CB^2 = PM^2 + QN^2$. Consequently, $CM^2 + CN^2 = CA^2 + CB^2$, by substitution or equal for equals.

These are the most material properties of the ellipsis; it remains now to say something of their areas, and the solids described by the rotation about either axes.

THEOREM.

313. *If from the center O of the ellipsis, with half of the first axis OA as radius, a quadrant of a circle ANC be described, then any ordinate PM to that axis is to that ordinate produced to the circle always in a given ratio of half the lesser axis to half the greater. I say, $PM : PN :: OB : OC$. Fig. 13.*

For the square of PN is, by the property of the circle, equal to the difference between the squares of OA and OP; and the square of PM, by the property of the ellipsis, to the difference between the squares of OA and OP, as the square of OB is to the square of OA or OC; therefore $PM^2 : PN^2 :: OB^2 : OC^2$. And consequently, $PM : PN :: OB : OC$.

314. If the ordinate PN be supposed to move from the position of OC towards A, or from A towards OC, by a parallel motion, the spaces OBMP : OCNP, or AMP, ANP, described by the parts PM, PN, of this line, will always be in the constant ratio of OB to OC: for the lines PM, PN, being constantly in that ratio, the spaces described at the same time, can be in no other.

315. If OM and ON are drawn, the triangles OMP, ONP, having the same base OP, are (*art.* 65) as their altitudes PM, PN, or as OB, OC, as has been proved above; these triangles are therefore in the same ratio as the segments AMP, ANP. The sectors OMA, ONA, the sums of these proportionals, are likewise in the same ratio of OB to OC.

316. Because the sector OAN, of the circle is (*art.* 175) equal to half the rectangle made by the arc AN and the radius OC; the elliptic sector OAM, which is to the former, as OC to OB, will be equal to half the rectangle made by the arc AN, and half the lesser axis OB.

OB. Hence the area of the whole ellipsis is equal to half the rectangle of half the lesser axis, and the circumference, whose radius is half the greater.

T H E O R E M.

317. *The spheroid described by the rotation of the ellipsis about the greater axis, is to the sphere of the circle described upon that axis as a diameter, as the square of half the lesser axis is to the square of half the greater.*

For because $PM:PN::OB:OC$, we have $PM^2:PN^2::OB^2:OC^2$; and as the areas of circles are as the squares (*art. 170*) of their radii, the circle described about OA, by PM, is to the circle described by PN, as the square of OB is to the square of OC; and as this always happens, the solid described by OBMP is to the solid described by OCNP, as the square of OB is to the square of OC; and so is also the semi-spheroid described by OBMA to the semi-sphere described by OCNA.

318. If the rectangle OD be made by the two semi-axes, and the diagonal OD intersects the ordinate PM in L; the difference between the squares of OB and PL, will be equal to (*art. 289*) the square of PM, or the difference between the circles of which they are radii, equal to the circle of the radius PM; and as this always happens, the difference between the cylinder described by the rectangle PB, and the cone described by the triangle OPL, will be equal to the solid described by the segment of the ellipsis PMBO. And consequently, the semi-spheroid is equal to the difference between the circumscribed cylinder and inscribed cone.

319. Hence it may be proved in the same manner as in *art. 247*, that the solid described by the sector OAM about the axis OA, is likewise the two thirds of the corresponding part of the circumscribed cylinder described by the rectangle PD.

S E C T III. H Y P E R B O L A.

D E F I N I T I O N S.

1. **I**F one end *f* of a ruler *fD* be fastened to a point, fix *f* in a plane, and a thread to another point,

fix F in the same plane, whose length differs from that of the ruler, by less than the distance f, F between these points fixt; and the other end of the thread fastened to the other end D of the ruler; then the pin M , which serves to keep the thread tight, and the part MD quite close to the edge of the ruler, will describe by the motion of the ruler about the point f , the Hyperbola MAE . Fig. 14.

By fastening the end of the ruler to the point F , and the thread to the point f , and retaining the same length, you may describe in the same manner as before, the opposite Hyperbola $Na n$.

2. The points fix F, f , are called the Foci.

3. The line Aa , passing through the foci, and terminated by the opposite hyperbolas, the Axis.

4. The point C which bisects the axis, the Center.

5. If Bb , perpendicular to the axis Aa , be taken so that the distances AB, Ab , are equal to that of the focus F from the center C , it is called the second axis.

The rest of the definitions of the hyperbola are the same as in the ellipsis.

320. The distance AF , from the focus to the vertex of the axis is equal to the distance af , of the other focus to the opposite vertex a ; for when the point M falls on A , the part FM of the thread becomes equal to AF ; and when the point m falls on a , the distance fm becomes fa ; and since FMD is equal to $fm d$ by definition, md is equal to MD , when the edge of these rulers coincide with the axis Aa ; and therefore $AF = af$.

321. Hence the distance Ff between the foci, is bisected by the center C ; for $CA = Ca$, and $AF = fa$; therefore $CF = Cf$.

322. The difference between the length of the ruler and that of thread is equal to the first axis Aa ; for when the edge of the ruler fD coincides with that axis, then will $fA - AF = fM - FM$, or, because $fA = fa + aA$, $aA = fM - FM$. Hence, if lines are drawn from the two foci f, F , to any point M in the hyperbola, the difference between these lines will be always to the first axis.

323. Lastly,

323. Lastly, the square of CB, half the second axis, will be equal to the rectangle AFa, or to Afa; for because AB is equal to CF, by defin. a F will be equal to the sum of the sides CA and AB, and AF equal to the difference; therefore (art. 46) $AFa = CB^2$.

T H E O R E M.

324. If from any point M, in the hyperbola, there be drawn a perpendicular MP to the first axis aA produced, and in that axis CX be taken so as $CA:CF::CP:CX$; then will $FM=AX$, and $fM=aX$.

If half the sum of the lines fM, FM, be call'd D, then because in the triangle fMF the rectangle of the sum 2D and difference 2CA, of the sides fM, FM, is equal to the (art. 46.) rectangle of the sum and difference of the segments fP, and FP: now their sum being fF, or 2CF, and their difference 2CP, we shall have $2D \times 2CA = 2CF \times 2CP$; or $CA:CF::CP:D$, whence $CX=D$ equal to half the sum, to which adding, or from which subtracting half their difference CA, we shall have aX for the greatest fM and AX for the least FM.

N. B. When the perpendicular MP falls within the triangle, fF is the sum, and CP half the difference of the segments fP, FP, and when that perpendicular falls without the triangle, fF is the difference, and 2CP the sum of the said segments.

T H E O R E M.

325. The square of any perpendicular MP to the first axis, is to the rectangle of the corresponding parts of that axis, as the square of half the second axis is to the square of half the first. I say, $PM^2:APa::CB^2:CA^2$.

Because CX or $CA + FM:CF::CP:CA$; subtracting antecedent from antecedent, and consequent from consequent, and comparing the terms of the second ratio to the differences $FM-AP:CP::AF:CA$; and comparing the terms of the last ratio to the sums of the antecedents and that of the consequents; I. $FM-AP::AF:CA$.

If in the first proportion we compare the terms of the second ratio to the sums of the antecedents and that
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of the consequents, $aP + FM : CP :: aF : CA$: lastly, comparing the terms of the last ratio, to the differences, between the antecedents and that of the consequents, II. $FM + FP : AP :: aF : CA$.

If the terms of the last proportion are multiplied by the respective terms of the third, and we take the square of PM for the rectangle of the (*art. 46*) sum and difference of the sides FM, FP, of the right angled triangle FPM, we get $PM^2 : APa : CB^2 : CA^2$, because (*art. 323*) $AFa = CB^2$. This is exactly the same as in the ellipsis, as well as all the succeeding demonstrations, a few signs excepted.

326. Since the square of any perpendicular PM drawn from any point of the hyperbola to the axis, is always to the rectangle of the corresponding parts aP, AP, of that axis, as the square of half CB, the second axis to the square of CA half the first ; it is evident that these perpendiculars on either side of the axis, equidistant from the center, are always equal. Whence, the first axis is also a diameter, whose ordinates MP are at right angles to it.

327. If the ordinate passes through the focus F such as FL, then the proportion $PM^2 : APa :: CB^2 : CA^2$ becomes $LF^2 : CB^2 :: CB^2 : CA^2$, because (*art. 323*) $CB^2 = AFa$, or (*art. 82*) $LF : CB :: CB : CA$. Therefore the ordinate which passes through the focus is the (*def. 6*) parameter to half the first axis.

328. Since (*art. 76*) $CA^2 : CB^2 :: CA : FL$, we have $PM^2 : APa :: FL : CA$, by equality of ratios. The square of any ordinate PM, is to the rectangle of the corresponding parts aP, AP, of the axis, as the parameter FL is to half the first axis CA.

329. If there be drawn another ordinate NQ to the first axis, then (*art. 325*) will $QN^2 : AQa :: FL : CA$, and as $PM^2 : APa :: FL : CA$, we shall have $QN^2 : AQa :: PM^2 : APa$, by equality (*art. 53*) of ratios ; or alternately, $QN^2 : PM^2 : AQa : APa$; or the squares of any two ordinates to the first axis, are as the rectangles of the corresponding parts of this axis.

330. The

330. The opposite hyperbolas $M A E$, $N a n$, are equal, so that if the one was laid over the other, they would agree in every point; for if $CQ = CP$, then will $AP = aQ$, and $AQ = aP$; therefore, $APa = AQa$, and $PM^2 = QN^2$; or, $PM = QN$. But this will happen every where, let the ordinates PM , QN , be drawn any where, provided they are equidistant from the center, they will always be equal. Therefore the opposite hyperbolas are equal in all respects.

331. Fig. 16. If from any point M in the hyperbola, a line MR be drawn perpendicular to the second axis Bb ; then because the rectangle APa is equal to (art. 46) the difference $CP^2 - CA^2$, of the squares of CP and CA , we have $PM^2 : CP^2 - CA^2 :: CB^2 : CA^2$; or because $CP = RM$, and $PM = CR$, that proportion will become $CR^2 : RM^2 - CA^2 :: CB^2 : CA^2$; and comparing the last ratio to that of the sums of the antecedents and of the consequents, $CB^2 + CR^2 : RM^2 :: CB^2 : CA^2$, or inversely, $RM^2 : CB^2 + CR^2 :: CA^2 : CB^2$. Which shews that the square of any ordinate RM to the second axis, is to the sum of the squares of half that axis and the distance CR of this ordinate from the center, as the square of half the first axis is to the square of half the second.

$P \quad R \quad O \quad B.$

332. To draw a line MD which shall touch the hyperbola in a given point M . Fig. 15.

From the foci f , F , draw two lines to the given point M ; in fM take ML equal to MF ; then the line MD , perpendicular to FL , will be the tangent required.

From any other point m , in mD , draw lines to the foci and to the point L ; then because $fm - Lm$, or its equal $fm - Fm$ is always less than (art. 23) fL , or its (art. 322) equal Aa , and as the difference between the lines drawn from the foci to any point in the hyperbola is always equal to the first axis; it is evident that the point m is not in the hyperbola. The line MD touches then the hyperbola only in the point M .

333. The angles fMD , FMD , made by the lines drawn from the foci to the point of contact M , are equal:

equal: for because $ML = MF$ by supposition, the perpendicular MD to the base FL of the isosceles triangle FML , will (*art. 18*) bisect the opposite angle M .

334. If the line CD be drawn, then because Cf is equal to CF , and FD equal to DL , the line CD will be (*art. 67*) parallel and equal to half the line fL ; that is, (*art. 332*) equal to CA ; if the tangent intersects the first axis in T and meets the second in t , then by the parallelism (*art. 67*) $fM : CD$, or $CA :: fT : CT$; or subtracting the consequents from their antecedents, and comparing the differences with the consequents, $fM - CA : fC :: CA : CT$: and if AX be taken equal to FM , then fM will (*art. 324*) be equal to aX , and $CA : CP :: CF : CX$; but $CX = fM - CA$: Therefore $CP : CA :: CA : CT$, by equality of ratios. *Which shews that the first semi-axis CA , is a mean proportional between the distances of the ordinate PM drawn from the point of contact and the point T , where the tangent meets that axis, from the center C .*

335. If the line Kk , drawn perpendicular to the tangent at M , meets the first axis in K , and the second in k , then because MK , LF , are parallel, we have (*art. 67*) $(fL)^2 CA : (fF)^2 CF :: fM : fK$; or, subtracting the terms of the first ratio from those of the second, $CA : CF :: fM - CA : CK$; but (*art. 324*) $CA : CF :: CP : CX = fM - CA$: therefore multiplying the terms of the last proportion by the respective ones of this, we get $CA^2 : CF^2 :: CP : CK$.

336. Hence, since $CA^2 : CF^2 :: CP : CK$; we get by subtraction, $CA^2 : CF^2 - CA^2$, or (*art. 323*) $CB^2 :: CP : PK$; and because (*art. 69*) $PK : PM :: pM$ or $CP : pk$; we have $CA^2 : CB^2 :: pk : PM$, or CP , by equality of ratios.

337. Because $CP \times CT : PK \times CT :: CP : PK$, we have $CA^2 : CB^2 :: CP \times CT : PK \times CT$; and as (*art. 334*) $CP \times CT = CA^2$; we have also $CB^2 = PK \times CT$: And therefore the similar triangles CTt , MPK , give $PM : PK :: CT : Ct$; and (*art. 85*) PM or $Cp : CB :: CB : Ct$, because $PK : CB :: CB : CT$. Which shews that the pro-

properties of the tangents are the same in respect to both axes.

N. B. We have hitherto followed the same steps, in the demonstrations of the several theorems relating to the hyperbola, as in the ellipsis; and what has been said in regard to one, may equally be applied to both. We might in the same manner proceed, were it not necessary to mention some lines peculiar to the hyperbola, and as the rest of the properties of the hyperbola, are more easily deduced from these lines than by the preceding method, we chose to follow this latter.

DEFINITION.

Plate XI. Fig. 17. The two lines nT , RS , drawn through the center C of the opposite hyperbolas, parallel to the lines Ab , AB , which join the extremities of the two axes Aa , Bb , are called Asymptotes.

338. Hence, the tangent EF , drawn through either extremity of the first axis, and terminated by the asymptotes, is equal to the second axis Bb , and bisected by the point of contact A : for since bC , AE , are parallel, as likewise bA and CE , the figure bE is a parallelogram; and therefore the opposite sides bC , AE , are equal; by the same reason the opposite side CB , FA , of the parallelogram FB , are equal. And $bB = EF$.

THEOREM.

339. *If a line RT , drawn through any point M in the hyperbola, parallel to either axes, be terminated by the asymptotes, the rectangle of its parts will be equal to the square of the semi-axes to which it is parallel.*

I. For by reason of the parallels AE and PT , we have (*art. 67*) $CA^2 : AE^2$ or $CB^2 :: CP^2 : PT^2$; and by the property of the hyperbola, in respect to the first axis, we have (*art. 325*) $CA^2 : CB^2 :: CP^2 - CA^2 : PM^2$; subtracting the terms of the last ratio in the last proportion, from the respective ones of the first, we get $CA^2 : CB^2 :: CA^2 : PT - PM^2$. Whence, $CB^2 = RMT$; because (*art. 46*) $PT^2 - PM^2 = RMT$.

II. Be-

II. Because AC is parallel to pr , we have $CA^2:AF^2$ or $CB^2::pr^2:Cp^2$; and by the property of the hyperbola, in respect to the second axis, we have (*art.* 331) $CA^2:CB^2::pM^2:CB^2+Cp^2$; and subtracting the terms of the last ratio in the first, from those of the last in the last, we get $CA^2:CB^2::pM^2-pr^2:CB^2$. Therefore $CA^2=rMt$; because (*art.* 46) $pM^2-pr^2=rMt$.

340. If there be drawn any where two parallels ad , Ll , the rectangles of their parts terminated by the asymptotes and the hyperbola, will be equal: through the points M , N , where these lines intersect the hyperbola, draw RT , Sn , parallel to one of the axis bB ; then, by reason of the parallelism, we have (*art.* 69) $MR:ML::NS:Na$, and $MT:ML::Nn:Nd$; multiplying the terms of one proportion by the respective terms of the other, we get $RMT:LML::SNn:aNd$; but $RMT=(\text{art. } 339) CB^2=SNn$. Therefore $LML=aNd$.

341. The parts ML , ul , of the same line drawn any how, terminated by the asymptotes and the hyperbola, are equal. For because (*art.* 340) $LML=lul$; or $LM \times Mu + ul = l \times uM + ML$; by subtracting the same rectangle $LM \times ul$, from both, we get $LM \times Mu = l \times uM$; or $LM=ul$, by division.

342. Fig. 18. Any tangent terminated by the asymptotes, is bisected by the point of contact; for since the parts Rm , MT , of any line RT , terminated by the asymptotes and the hyperbola, are always equal, when that line becomes the tangent EF , the points M , m , coincide with the point of contact A . Hence, $AF=AE$, and the rectangle TMR or TmR , is always equal to the square of half that tangent AE or AF .

343. The line Mm , parallel to the tangent $F.F$, and terminated by the hyperbola, is bisected by the line CP , drawn through the center C and the point of contact A ; for since $AF=AE$, PT will be (*art.* 67) $=PR$; and as $Rm=MT$, it is evident that $PM=Pm$. Whence, aA is a diameter, and Mm its ordinate,

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Which shews that all lines passing through the center, and meet the hyperbolas, are diameters.

DEFINITION.

If from the extremity A of a diameter, two lines AB, Ab, are drawn parallel to the asymptotes, so as meet a line Bb, parallel to the tangent at A; this line Bb, is said to be the conjugate diameter to the diameter Aa.

344. It follows from this definition, that the conjugate diameter Bb, is equal to the tangent EF to which it is parallel; since the opposite sides bC, AE and FA, CB, of the parallelograms bE, FB, are equal.

THEOREM.

345. *The square of an ordinate PM to any diameter Aa, is to the rectangle of the corresponding parts of that diameter, as the square of half its conjugate is to the square of half that diameter. I say, $PM^2 : APa :: CB^2 : CA^2$.*

Because (art. 339) TMR , or (art. 46) $PT^2 - PM^2 = AE^2$; we have $PT^2 - AE^2 = PM^2$; and the similar triangles CAE, CPT, give $PT^2 : AE^2 :: CP^2 : CA^2$; or, (art. 75) $PT^2 - AE^2 : CP^2 - CA^2 :: AE^2 : CA^2$. But $PT^2 - AE^2 = PM^2$; $CP^2 - CA^2 = APa$, and $AE^2 = CB^2$. Consequently, $PM^2 : APa :: CB^2 : CA^2$.

346. Since the square of an ordinate PM, to any diameter, is always to the rectangle of the corresponding parts of the diameter, as the square of half its conjugate is to the square of half that diameter, whether the diameter be an axis or not; it is evident, that whatever has been deduced from that property, in regard to the axes, will also be true in regard to any diameters.

THEOREM.

347. *If a line mM intersects any diameter Nn, either within or without the hyperbola, the rectangle of its parts, terminated by the hyperbola and this diameter, will be to the rectangle of the corresponding parts of that diameter, as the square of half the diameter parallel to that line is to the square of half the intersecting diameter. I say, $mSM : NSn :: CB^2 : CN^2$.*

By

By reason of the parallels LQ, TP, we have CQ : CP or CN : CS :: QL : PT :: QN : PS; and therefore the squares of these lines are also proportional, as likewise the squares of the terms CN, CS, are proportional to the differences between the squares of QL, QN, and PT, PS; but the difference between the squares of QL, QN, is equal to the square of (art. 342) CB; whence the square of CN is to the square of CS, as the square of CB is to the difference between the squares of PT and PS: And if we subtract the antecedents from their consequents, the square of CN is to the difference between the squares of CS and CN, or the rectangle NSn, as the square of CB is to $PT^2 - CB^2 - PS^2$; but the difference between the squares of PT and CB (art. 339) is equal to the square of Pm. Consequently, $CN^2 : NSn :: CB^2 : MSm$; because (art. 46) $Pm^2 - PS^2 = mSM$, or alternately and by inversion, $mSM : NSn :: CB^2 : CN^2$.

348. If any other line Kk, parallel to the semi-diameter CX, meets the diameter nN, in the same point S, as Mm, and the hyperbola in K, k; then will $(CN^2 : NSn ::) CB^2 : MSm :: CX^2 : kSK$; or alternately, $CB^2 : CX^2 :: MSm : kSK$.

349. If the line Hh, parallel to Mm, meets kK, produced if necessary, in D; it is evident that $(CB^2 : CX^2 ::) MSm : kSK : HDh : KDk$.

In general, the rectangles of the parts of two parallel lines, terminated by the hyperbola and any other line, are always as the rectangles of the corresponding parts of that line, terminated by the hyperbola and these parallels; and if one or two of these lines become tangents, the rectangles of their parts will become the squares of those lines. This is a natural consequent from the foregoing corollaries.

THEOREM.

350. If from any point N in the hyperbola two lines, NQ, NV, are drawn parallel to the asymptotes, and being terminated by the same; their rectangle will be always the same, let that point be taken where it will. Fig. 17.

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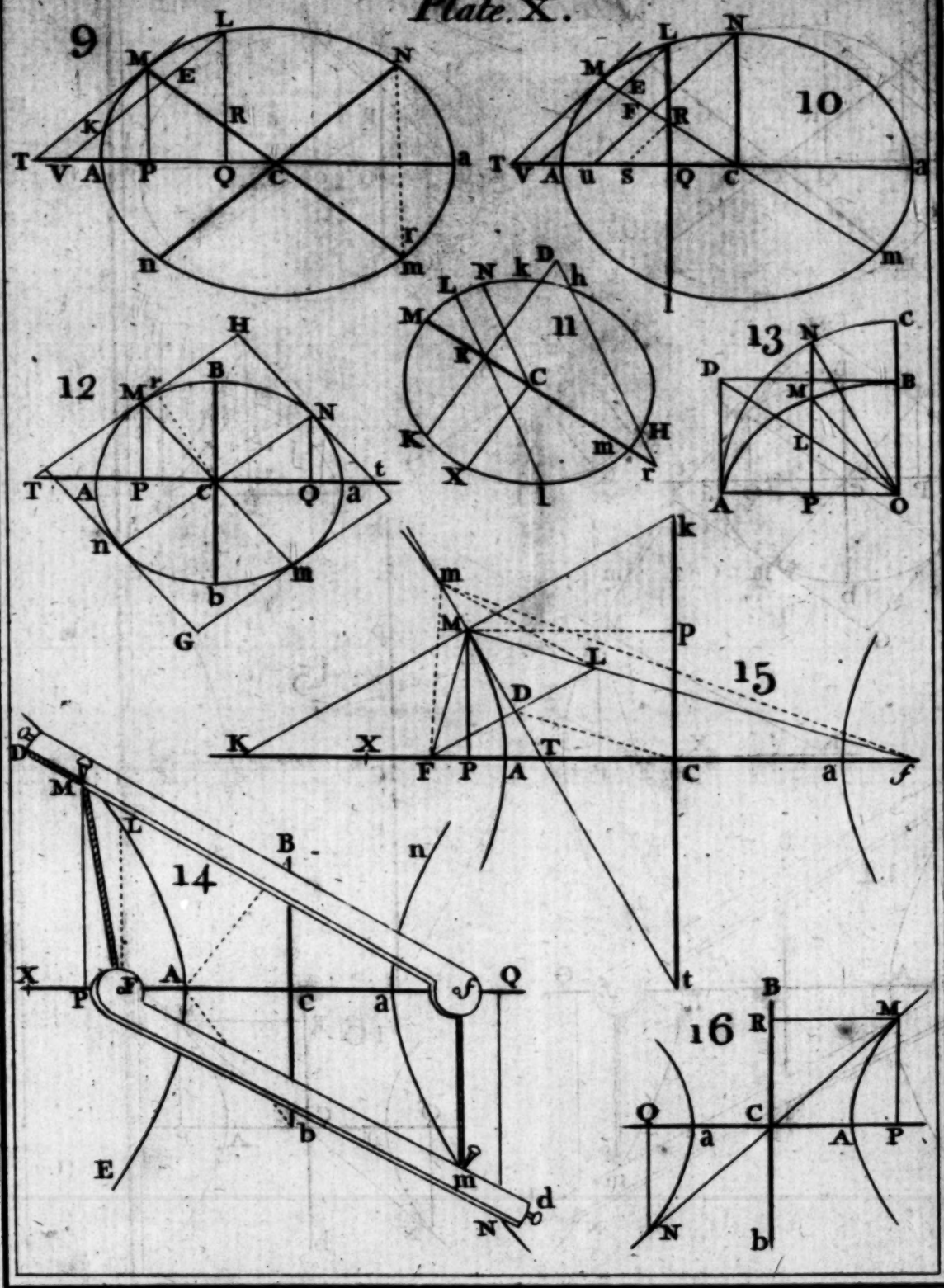
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Let AB intersect the asymptote in H, then both triangles QNS, NVn, will be similar to the triangle CHB, as having all their sides parallel; and therefore $NS:NQ::CB:CH$, and, $Nn:NV::CB:HB$; multiplying the terms of one proportion by the respective terms of the other, we get $SNn:NQ \times NV::CB^2:CH \times HB$; but (art. 339) $SNn=CB^2$. Consequently $NQ \times NV=CH \times HB$.

351. Since the opposite sides of the figure VQ are parallel, it is a parallelogram, and so the opposite sides VN, and CQ are equal; and because CH is parallel to bA, and $bC=CB$, AH is also equal to HB; and therefore $CQ \times QN=CH \times HA$. Which shews, that if from any two points in one asymptote, two lines are drawn parallel to the other; and being terminated by the hyperbola, the rectangle of one part of the asymptote, and the corresponding line, will be equal to the rectangle of the other part and its corresponding line.

T H E O R E M.

352. The parallelogram GH, made by any two conjugate diameters Mm, Nn, is equal to the rectangle made by the two axes Aa, Bb. Fig. 19.

Let the lines AB, MN, joining the extremities of the axes and conjugate diameters, intersect the asymptote in L and R; then because $CL \times LA=CR \times RM$, we have $CL:CR::RM:LA$, or $CL:CR::NM:AB$; therefore the triangles BAC, NCM, which have their bases and altitude reciprocally proportional, are (art. 93) equal, as well as their octuples, the parallelogram GH, and the rectangle made by the two axes.

T H E O R E M.

353. The difference between the squares of any two conjugate diameters CM, CN, is equal to the difference between the squares of the two axes CA, CB.

Let Cb be perpendicular to MN, intersecting AB in a; then the similar triangles CLa, CRb, give $CL:CR::La:Rb$; and by the property of the asymptotes $CL:CR::NM:BA$; therefore $La:Rb::$

K

NM:

NM : BA ; or $L a \times AB = R b \times NM$. Hence, because the rectangle of the sum and difference between the segments Mb, bN, of the triangle CNM, is equal to the rectangle of the sum and difference of the segments Aa, aB, of the triangle CBA ; their equals (*art. 50*) the rectangles of the sum and differences of the sides CN, CM, and CB, CA, are also equal ; and consequently, the difference between the squares of CM, CN, will be (*art. 46*) equal to the difference between the squares CA and CB.

T H E O R E M.

354. *The spaces BMQS, FGRV, terminated by two parallel lines VQ, RS, the asymptotes and the hyperbola, will be always equal.* Fig. 20.

For since the parts BS, RG, of the same line RS, terminated by the asymptotes and the hyperbola are (*art. 341*) always equal, it is evident, that if this line be supposed to move parallel to itself, the equal parts BS, RG, will describe equal spaces at the same time ; and therefore when RS is moved to any situation VQ,

space BMQS, described by BS, will be equal to the space FGRV, described by RG.

355. If the lines BA, MP, are drawn parallel to the asymptote CV, and GD, FE, parallel to the asymptote CQ, the spaces ABMP, DGFE, terminated by these parallels, the asymptotes and the hyperbola are equal: For the triangles BAS, RDG, having the sides BS, RG, (*art. 341*) equal, and all their sides parallel, are similar and (*art. 25*) equal ; and by the same argument the triangles MPQ, VEF, are equal. By adding the equal triangles BAS, DRG, to, and subtracting the equal ones MPQ, VEF, from the equal spaces SBMQ, GRVF, we shall have the space ABMP equal to the space DGFE.

356. If from the points F, G, the lines FL, GK, are drawn parallel to the asymptote CV, meeting the other CQ in L, K ; then because (*art. 350*) $CD \times DG$ is equal to $CE \times EF$, the rectangles CG, CF, are equal ; by subtracting the rectangle CT which is common to both,

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both, the remainders LG, TE, are equal, and adding to these equals the same space TGF, the sums or spaces LKGF, DGFE, are equal; and since the space DGFE has been proved (*art. 355*) to be equal to the space ABMP, the space LKGF is also equal to the space ABMP.

357. If from the center C lines are drawn to the points B, M, G, F, the sector CBM will be equal to the corresponding space ABMP, and the sector CGF, equal to the space DGFE: For since (*art. 350*) $CA \times AB = CP \times PM$, the triangles CAB, CPM, are (*art. 93*) equal; from which taking the common space CAH, and adding to the remainders, the curvilinear space HBM, the sector CBM, will be equal to the space ABMP.

And by the same argument the sector CGF is equal to the space DGFE.

358. Hence, the parts CL, CK, CA, CP, of the same asymptote, terminated by the lines which include equal spaces, are proportional: For since (*art. 69*) $EF : DG :: VF : RG$, and $MP : AB :: MQ : BS$; and because the parts FV, MQ, are (*art. 69*) equal as well as RG, BS, we have $EF : DG :: MP : AB$, by equality of ratios; and as CL is equal to EF, and CK to DG, the last proportion becomes $CL : CK :: MP : AB$; but (*art. 350*) $MP : AB :: CA : CP$: Therefore $CL : CK :: CA : CP$, by equality of ratios.

359. If $\therefore CL : CK : CA$, the space KGFL will be equal to the space KGBA, or the sector CGF, equal to the sector CBG: For if in the former proportion CA becomes CK, and CP, CA, the space ABMP will become the space KGBA, and still remain equal to the space KGFL; since the line MF becomes FB in this case, and BG a tangent at G parallel to FB.

If $\therefore CL : CK : CA : CP$, it is evident from the preceding corollary, that the spaces LG, KB, AM, are equal; or the space LFGK is to the space LFMP as unity to 3: For since $\therefore CL : CK : CA$, the space LFGK is equal to the space KGBA; and as $\therefore CK : CA :$

CA:CP, the space KGBA is equal to the space ABMP; therefore these three spaces are equal.

It may be proved in general, that if there be any number m of continual geometrical proportionals, such as CL, CK, CA, CP, &c. the lines drawn to the points L, K, A, P, parallel to the asymptote CV, will include $m-1$ equal spaces; and therefore the first LG, will be to the sum of them all, as unity is to the number $m-1$.

From this last corollary arises an easy and short method for computing logarithms; and notwithstanding that some think they should be deduced from the properties of numbers, we shall here explain their nature and the manner of computing them; in this we follow Sir *Isaac Newton*, as may be seen in his Works translated by Mr. *Colson*; and the rather, on account that those who pretend to deduce them from the properties of numbers, are obliged to make use of so far-fetched principles as I, and I believe a great many, could never understand.

DEFINITION.

The logarithms of numbers, which are in a geometrical progression, are such numbers as are in an arithmetical progression; or, if CL, CK, CA, CP, express any numbers, and CL being unity; the spaces LG, LB, LM, will express the logarithms of these numbers CK, CA, CP: And if CA expresses unity, the space AM expresses the logarithm of any number CP greater than unity, and AG, the logarithm of any number less than unity; and therefore the logarithm of unity is nothing.

The tables of logarithms contain the natural numbers from unity to 10,000 or to 100,000, and against them stand their logarithms; and by the nature of the logarithms the sum of the logarithms of any two numbers will be that of their product; the difference that of their quotient; and half, or one third of the logarithm of any number, will be that of its square or cube root; and

and lastly, twice or thrice the logarithm of any number, will be that of the square or cube of this number.

Hence, instead of multiplying numbers together, you only add their logarithms, and the sum is the logarithm of their product: To divide a number by another, you subtract the logarithm of the divisor from that of the dividend, and the difference is the logarithm of the quotient.

Instead of extracting any root of a number, you divide the logarithm by the exponent of the power; or, instead of squaring or cubing any numbers, you take the double or triple of its logarithm, and against the quotient or product the root or power is found.

P R O B L E M.

360. *To find the space AM, or which is the same, to find the logarithm of any given number CP, greater than unity. Fig. 20.*

Let GBM be an equilateral hyperbola, so that the asymptotes CV, CP, may be at right angles to each other, and let CA express unity; AB any number a ; and AP, x ; so that CP, or $1+x$, expresses the given number: Now as the rectangle of CA, and AB (a) is always equal (*art. 350*) to the rectangle of CP and PM, we have $a = 1+x \times PM$; or dividing by $1+x$,

$$PM = \frac{a}{1+x} : \text{If this value of the perpendicular PM,}$$

be reduced into an infinite series by a continual division, we shall have $PM = ax$ by $1-x+xx-x^3+x^4-\&c.$ and therefore the area ABMP will be (*art. 233*) $= ax$ by $x - \frac{1}{2}xx + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \&c.$ or this value will be the logarithm of the number $1+x$.

If the logarithm or area, ABGK, of any number CK less than unity be required, by supposing $AK=x$;

then by what has been said $KG = \frac{a}{1-x}$, and by the same argument as before $a \times$ by $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \&c.$

+ $\mathcal{E}c$. will express the area AG, or the logarithm of the number CK less than unity. It must be observed, that as the logarithm of unity is nothing, and that of any number greater than unity positive, that of any number less than unity must be negative; and since x cannot be greater than unity otherwise the logarithms above become impossible: We must find an expression, by which the several logarithms may be found, one after another.

361. If the logarithm of $1-x$ be subtracted from the logarithm of $1+x$, we shall have the logarithm of $\frac{1+x}{1-x}$, by the nature of logarithms, but as the logarithm of $1-x$ is negative, instead of subtracting, it must be added, which gives $2ax$ by $x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \mathcal{E}c$. for the logarithm of $\frac{1+x}{1-x}$; or, if $A=2ax$, $B=xxA$, $C=xxB$, $D=xxC$, $\mathcal{E}c$. this expression will become $A + \frac{1}{3}B + \frac{1}{5}C + \frac{1}{7}D + \mathcal{E}c$.

As the number a , or the rectangle AD may express any number, so there may be various sorts of logarithms, when $a=1$, then the logarithms are called hyperbolic, or natural.

It has been found convenient in practice, that the logarithm of 10 should be unity, and the common tables have been constructed accordingly; therefore the number expressed by a must be found, which may be done by finding the natural logarithm of 10 in the following manner.

Let n express the logarithm of $\frac{5}{4}$, and m that of $\frac{128}{125}$; then I say, that $10n + 3m$ will be the natural logarithm of 10; for if x be the logarithm of 2, and y that of 5: then because $\frac{5}{4} = \frac{5}{2 \times 2}$; the logarithm y of 5, minus twice the logarithm x of 2, will be equal to the logarithm n of $\frac{5}{4}$; that is, $y - 2x = n$; and since $\frac{128}{125} = \frac{2^7}{5^3}$ se-

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ven times the logarithm x of 2; minus three times the logarithm y of 5, will be equal to the logarithm m of $\frac{128}{125}$; that is, $7x - 3y = m$; this will appear evident from what has been said above, for the logarithm of any power of a number is equal to the logarithm of that number multiplied by the exponent of the power; and the logarithm of the quotient of one number divided by another, is equal to the difference between the logarithm of the divisor and the dividend.

Now if $y - 2x = n$ be multiplied by 3, it will become $3y - 6x = 3n$, which being added to $7x - 3y = m$, gives $x = 3n + m$; and since $y - 2x = n$, gives $y = n + 2x$, by transposition, and $x = 3n + m$, multiplied by 2, gives $2x = 6n + 2m$; the sum of this and $y = n + 2x$, gives $y = 7n + 2m$: and because 2×5 is 10, the sum of the logarithms x and y , gives $x + y = 10n + 3m$, for the logarithm of 10.

E X A M P L E I.

362. To find the logarithm of $\frac{1+x}{1-x}$.

Suppose $\frac{1+x}{1-x} = \frac{5}{4}$; by multiplying crosswise $4 + 4x = 5 - 5x$, or $9x = 1$ by transposition; hence $x = \frac{1}{9}$; by writing the value of x into those of A, B, C, &c. we get $A = \frac{2}{9}$, $B = \frac{A}{81}$, $C = \frac{B}{81}$, $D = \frac{C}{81}$, &c. which being reduced into decimals gives,

$$\begin{aligned} .22222, 22222, 22222 &= A \\ 274, 34842, 249657 &= B \\ 3, 38701, 756168 &= C \\ 4181, 503162 &= D \\ 51, 623496 &= E \\ 637327 &= F \\ 7868 &= G \\ 97 &= H, \end{aligned}$$

And $A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D +, \&c.$ gives

22222,22222,22222

91,44947,416552

67740,351234

597,357595

5,735944

57938

606

6

22314,35513,142097 = n.

And 2.23143,55131,42079 = 10n.

E X A M P L E II.

363. To find the logarithm of $\frac{128}{125}$.

Suppose $\frac{128}{125} = \frac{1+x}{1-x}$; by reducing this equation we

get $x = \frac{3}{253}$; and $A = \frac{6}{255}$, $B = \frac{9A}{253}$, $C = \frac{9B}{253}$,

$D = \frac{2C}{253}$; these numbers, being reduced into decimals, give

.02371,54150,197628 = A

33345,113215 = B

4,688497 = C

659 = D

And .02371,54150,197628

11115,037738

937699

94

.02371,65266,173159 = m,

And .07114,95798,519477 = 3m

2.23143,55131,420970 = 10n

2.30258,50929,94045 = 10n + 3m = log. 10.

As

As this number multiplied by a , is equal to unity, or to the tabular logarithm of 10, by dividing unity by this number we shall have $a = 0.434294481903252$.

Having thus found the value of a , the tabular logarithms are found as follows.

E X A M P L E III.

364. To find the logarithm of 9.

Suppose $\frac{1+x}{1-x} = \frac{10}{9}$, then will $x = \frac{1}{19}$, $A = \frac{2a}{91}$,

$B = \frac{A}{19}$, $C = \frac{B}{19}$, &c. and therefore these numbers, reduced into decimals, give

$$\begin{aligned} .04571,52086,21395 &= A \\ 12,66349,26926 &= B \\ 3507,89271 &= C \\ 9,71715 &= D \\ 2691 &= E \end{aligned}$$

$$\begin{aligned} .04571,52086,21395 \\ 4,22116,42309 \\ 701,57856 \\ 1,38816 \\ 299 \end{aligned}$$

$$.04575,74905,60675 = \log. \frac{10}{9}.$$

As the logarithm of 10 is unity, and 10 divided by $\frac{10}{9}$ gives 9, the logarithm of $(\frac{10}{9})$, just now found, subtracted from unity, gives 0.95424,25094,39325 for the logarithm of 9.

And because the square root of 9 is 3, half the logarithm of 9 gives 0.47712,12547,19662, for the logarithm of 3.

E X A M P L E IV.

365. To find the logarithm of 8.

Suppose $\frac{1+x}{1-x} = \frac{81}{80}$; then will $x = \frac{1}{161}$, $A = \frac{2a}{161}$,

$B = \frac{A}{161}$, $C = \frac{B}{161}$: And $A + \frac{1}{3}B + \frac{1}{3}C$, gives .00539,

$$\begin{array}{r} .00539,49625,08115 \\ 693,76985 \\ \hline 1604 \end{array}$$

$$.00539,50318,86704 = \log. \frac{81}{10}$$

Since $9 \times 9 = 81$, and 81 divided by $\frac{81}{10}$ gives 80, the logarithm of $\frac{81}{10}$, subtracted from twice the logarithm of 9, found above, gives $.90308,99869,91946$, for the logarithm of 80; and as 80, divided by 10, gives 8, taking unity the logarithm of 10, from that of 80, we shall have $0.90308,99869,91946$, for the logarithm of 8.

As $2 \times 2 \times 2 = 8$; one third of the logarithm of 8 gives $0.30102,99956,63982$, for the logarithm of 2.

Because $2 \times 3 = 6$, by adding the logarithm of 2 and 3, we shall have $0.77815,12503,83644$, for the logarithm of 6.

E X A M P L E V.

366. To find the logarithm of 7.

Suppose $\frac{1+x}{1-x} = \frac{49}{48}$; then will $x = \frac{1}{97}$, $A = \frac{2a}{97}$

$B = \frac{A}{97}$, $C = \frac{B}{97}$; and $A + \frac{1}{3}B + \frac{1}{3}C$, gives

$$\begin{array}{r} .00895,45254,00067 \\ 3172,32628 \\ \hline 29228 \end{array}$$

$$.00895,48426,52923 = \log. \frac{49}{48}$$

Because $6 \times 8 = 48$, and 48 multiplied by $\frac{49}{48}$, gives 49; the logarithm of $\frac{49}{48}$ being added to the sum of the logarithms of 6 and 8 found above, gives $1.69019,60800,28513$, for the logarithm of 49; or because $7 \times 7 = 49$, half this log. $0.84509,80400,14256$ will be the logarithm of 7.

These examples are sufficient to shew the method of computing the logarithms of all the numbers; and it may be observed, that the logarithms of large numbers are found by one step, or two at most; or are found only

only by addition and subtraction: For the logarithms of even numbers, or which are the product of any number of factors, are found by adding those of their factors; so that those of the prime numbers are only to be found by the series above; and this is easily done by having the logarithms next under and above them; for, if any three numbers exceed each other by unity, the square of the mean will always exceed the product of the two extremes, by unity: Thus, to find the logarithm of 11, the next prime number; the square of 11 is 121, and the product of 10 and 12 is 120; then finding the logarithm of $\frac{121}{120}$, to which adding the logarithm of 10×12 , and half the sum will be the logarithm of 11.

To find the logarithm of 13, the square of which is 169, and the product of 12 by 14 is 168; then the logarithm of $\frac{169}{168}$, added to the logarithm of 12×14 ; half this sum will be the logarithm of 13; in this manner the logarithm of any prime number is found.

It must be observed, that when the logarithm of any fraction is to be found, the value of x is always expressed by a fraction; whose numerator is the difference, and denominator the sum of the numerator and denominator of the given fraction.

Thus, if $\frac{1+x}{1-x} = \frac{121}{120}$, then will $x = \frac{1}{241}$, and if

$\frac{1+x}{1-x} = \frac{289}{288}$; then will $x = \frac{1}{577}$.

As the right use of the tables of logarithms is of great importance, we shall explain, in a few words, the most remarkable difficulties: First, when the logarithm of any number is to be found greater than any in the tables; when a logarithm is given, and the number answering to it, is not to be found exactly; or to find the logarithms of vulgar and decimal fractions; we shall explain those four cases one after another.

E X A M P L E I.

367. To find the logarithm of any given number, not exceeding the number of places, to which the logarithms are computed.

The

The least tables are computed to 10.000; so if the logarithm of any number less than 10.000 is wanted, it may readily be found in the tables; thus the logarithm of 7987, will be found opposite to that number in the tables to be 3.9023837.

Before we find the logarithm of any number of above five places, it must be observed, that the index, or characteristic, of the logarithms of one place, or under 10, is 0; that of two places, or between 10 and 100, is 1; that of three places, 2; that of four, 3; and so on; that is, the index is always unity less than the number of places of the given number.

To find the logarithm of a number, as 56789: I divide it by 10; which gives 5678.9.

Because the logarithm of 5678 is .7541954, omitting the index; and that of 5679, is .7542719; as the first is less and the second greater, than the logarithm wanted, take their difference, which is 765; take likewise the difference between the least number 5678, and the given number 5678.9, which is .9; multiply these differences together, and add the product 688.5, or only 688, to the logarithm of 5678, and the sum .7542642, will be the logarithm sought, by prefixing the index 4 to it.

The same logarithm will be found by multiplying the difference 765, by the difference .1, between the given number and the greater, and subtracting the product 76.5, from the logarithm of the greater. Hence follows this

GENERAL RULE.

To find the logarithm of any number greater than 10.000.

Strike off the figures above four, as decimals 5678.9. take the difference 765, between the logarithms of the numbers next above and below the given one.

Multiply this difference by the difference .9 or .1, between the given number and that next above or below it.

Add

Add the product to the least, or subtract it from the greatest, then the sum or difference will be the logarithm sought.

E X A M P L E II.

368. *To find the number of a given logarithm.*

If the logarithm is found exactly in the tables, the number will be opposite to it; but if it cannot be found, observe the following

G E N E R A L R U L E.

Take the difference between the logarithms next above and below the given one.

Take likewise the difference between the given logarithm and one of the others.

Divide this last difference by the former; and add or subtract the quotient to the former number encreased by as many cyphers as the quotient contains places, according as the logarithm taken is less or greater than the given logarithm.

Let the given logarithm be	- -	3.7890423
Then the logarithm of 6152 is less	-	7890163
And that of 6153 is greater	- -	7890869
The first difference is	- - - - -	706
And the second	- - - - -	260
Therefore the number sought is	-	6152.367

E X A M P L E III.

369. *To find the logarithm of any vulgar fraction.*

Subtract the logarithm of the denominator from that of the numerator, and the difference will be the logarithm wanted.

Thus the logarithm of $\frac{7}{5}$ is equal to the difference .1461280, between the logarithms .8450980 of 7, and that .6989700 of 5.

But if the fraction is proper, that is, if the numerator is less than the denominator, the numerator must be multiplied by 10, or 100, so as to become greater than the denominator; then subtract from the index as many units as there has been cyphers added, thus the logarithm of $\frac{3}{4}$ by multiplying by 10, to get $\frac{30}{4}$,
is

is .8750613; and subtracting unity, -1.8750613 ; that of $\frac{3}{8}$; or $\frac{300}{800}$ is 0.5740313; and subtracting 2 gives -2.5740313 for the logarithm required.

The logarithm of $\frac{3}{4}$, or of $\frac{300}{400}$ is 0.7632109, and subtracting 3, -3.7632109 .

E X A M P L E IV.

370. *To find the logarithm of any decimal fraction.*

Consider decimals as whole numbers, and subtract as many units from the index as there are decimal figures.

Thus, the logarithm of .125, considered as whole numbers, is 2.0969100; and subtracting 3 from the index gives -1.0969100 .

The logarithm of .00126, by considering 126 as whole numbers, is 2.1003750; and subtracting 5 from the index gives -3.1003750 .

The logarithm of any mixt number is also found in the same manner, by considering them as whole numbers, and subtracting as many units from the index as there were decimals; or prefixing the index of the whole number as if there were no decimals. Thus, the logarithm of 2.134, considered as whole numbers, is 3.3291944; and subtracting 3 from the index gives .3291944; this appears from the numbers also, since there is but one figure, namely 2, of whole numbers; the index must by course be nothing.

Having given the manner of constructing the logarithms, as likewise how to find them in the tables, so as to answer any given numbers, the application of them will be shewn in the next section; so there remains now to determine the solid contents of hyperboloids.

P R O B L E M.

371. *To find the content of the solid described by the segment AMP, about the first axis CP. Fig. 21.*

Draw the asymptote CN, the tangent AE, and EL, parallel to the axis intersecting the ordinate PM in L; and let that ordinate produced meet the asymptote in N.

Then because AE or PL is equal to half the second axis, and (art. 339) AE^2 or $PL^2 = PN^2 - PM^2$ or, by transposition,

transposition, $PM^2 = PN^2 - PL^2$; and since the areas of circles are (*art.* 170) as the squares of their radii, the difference between the circles described by PN and PL, or the area described by LN, in the rotation of the figure, will be equal to the area described by the ordinate PM; and as this always happens, the solids described by the triangle ELN, will be equal to the hyperboloid described by the corresponding segment APM.

Which shews, that the hyperboloid described by any segment AMP, is equal to the difference between the frustrum of the cone, described by the space AENP, and the cylinder described by the rectangle AELP.

B O O K III.

Of PLANE TRIGONOMETRY.

SECT. I. DEFINITIONS.

1. **T**RIGONOMETRY is the science which considers the sides and angles of triangles, and is divided into plane and spheric; the plane considers right lined triangles; and the spheric, those which are formed on the sphere by its great circles. It is only the former which we propose to treat of, the other being of no use to military gentlemen, for whom this work is chiefly intended.

2. In order to solve the different problems in trigonometry, it is necessary not only that the circumference of a circle be divided into degrees and minutes, as has been shewn in the sixth section of the first book, but likewise certain lines within and without the circle are to be expressed by parts of the radius.

3. Plate XII. Fig. 1. The sine of an arc is the line drawn from one of its extremities, perpendicular to the diameter which passes through the other: Thus if PM be perpendicular to the diameter Aa, this line PM is the sine of the arc AM; and because the arc Ma is likewise determined by the same line PM, it is also

also the sine of the supplement Ma to the arc AM .

4. The cosine of an arc AM is the part CP of the diameter terminated by the center C and the sine PM of that arc.

If the radius CB be drawn at right angles to the diameter Aa , and MQ perpendicular to it; then because CP and QM are equal, and QM being the sine of the arc MB ; the cosine CP of an arc AM , is equal to the sine of its complement MB .

5. The versed sine of an arc AM , is the part AP , of the diameter terminated by its sine and that arc.

6. The tangent of an arc AM , is the line AT , which touches it at one extremity A , and is terminated by a line drawn from the center C through the other M .

7. The secant is the line CT drawn through the center, and terminated by the tangent.

8. Lastly, the cotangent and cosecant of an arc AM , are the tangent Bt , and secant Ct , of its complement BM .

There are tables which contain the sines, tangents, and secants of every arc, from a minute to 90 degrees, expressed by parts of the radius divided into 10000000 equal parts; the principles on which their computation depends are contained in the following propositions.

P R O B L E M.

372. *Any arc AM being given or expressed by parts of the radius, to find its sine PM . Fig. 1.*

If the arc AM be called z , and its sine PM , x ; then because (art. 252) $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots$, &c. by cubing each side we get $z^3 = x^3 + \frac{1}{2}x^5$, neglecting the higher powers, and $z^5 = x^5$; the first divided by 6 gives $\frac{1}{6}z^3 = \frac{1}{6}x^3 + \frac{1}{12}x^5$; and the second by 120, $\frac{1}{120}z^5 = \frac{1}{120}x^5$. These values of x and its powers being substituted into the value of the arc, gives $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5$, nearly.

The higher powers of x and z have been neglected, because the expression of the sine is sufficiently exact, as will

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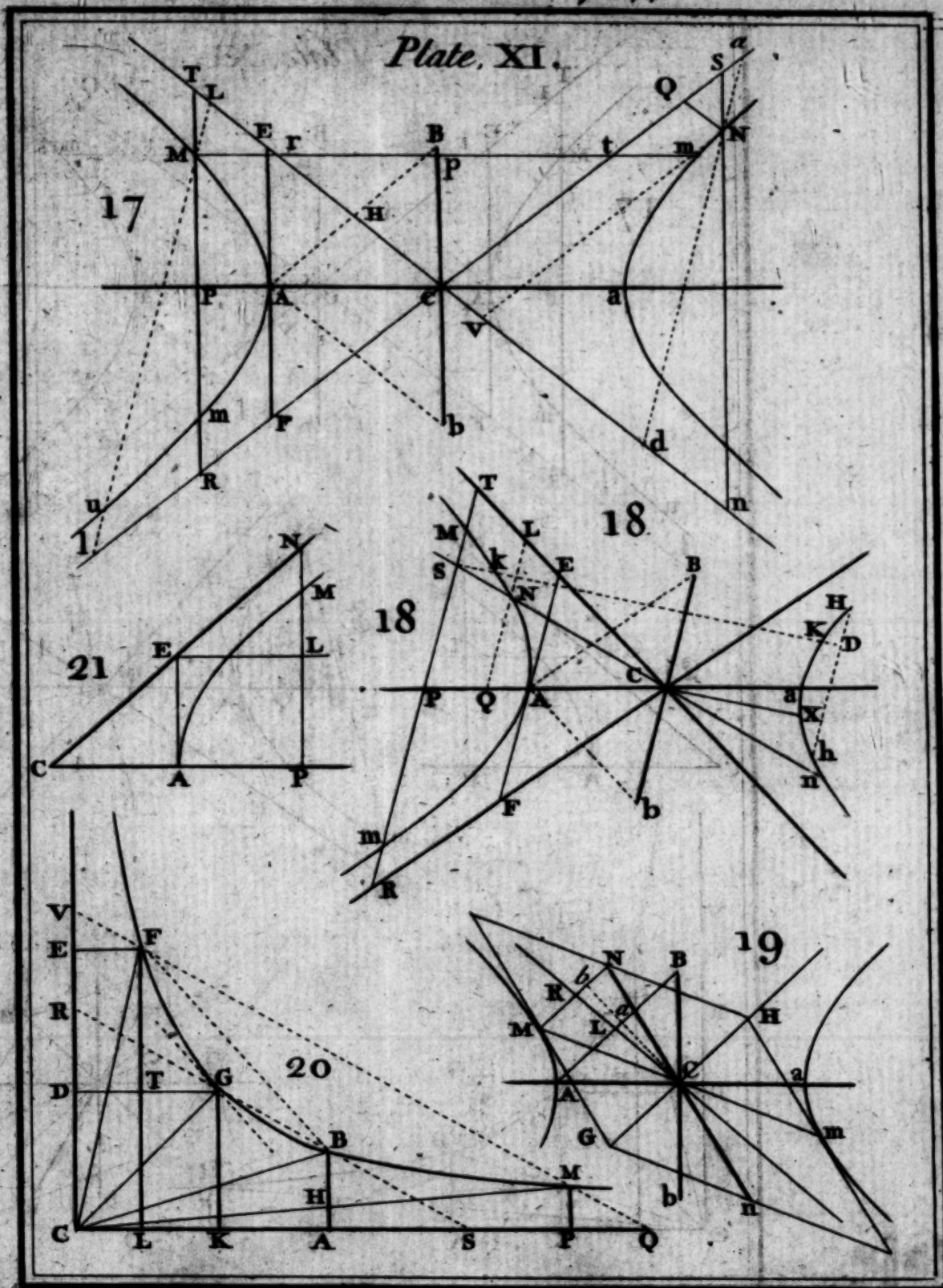
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will appear hereafter: for, when the arc is but small, the difference between the arc and the sixth part of its cube, will give the sine of that arc nearly.

P R O B.

373. *Any arc AM being given or expressed by parts of the radius, to find the cosine CP of that arc.*

Let the cosine CP be called v ; then because of the right angled triangle CPM, the difference between the squares of CM (1) and PM (x) will (art. 48) be equal to the square of the cosine; that is, we have $vv = 1 - xx$, or $v = \sqrt{1 - xx}$: and the square root of $1 - xx$, by the rule given in algebra, will be $v = 1 - \frac{1}{2}xx - \frac{1}{8}x^4 - \frac{1}{16}x^6$ &c. Now because we have (art. 372) $x = z - \frac{1}{8}z^3 + \frac{1}{64}z^5 - \frac{1}{256}z^7$ &c. the square of which is $xx = zz - \frac{1}{4}z^4 + \frac{1}{16}z^6 - \frac{1}{64}z^8$ &c. by neglecting the higher powers. These values of xx and x^4 , being substituted into $v = 1 - \frac{1}{2}xx - \frac{1}{8}x^4$, give $v = 1 - \frac{1}{2}zz + \frac{1}{8}z^4$ nearly for the cosine CP required.

It is evident that the versed sine AP is equal to (CA - CP) $\frac{1}{2}zz - \frac{1}{8}z^4$ nearly.

374. If the tangent AT be called t ; then because the cosine CP is to the sine PM, as the radius CA is to the tangent AT; it is evident, that the sine PM ($z - \frac{1}{8}z^3 + \frac{1}{64}z^5$) divided by the cosine CP ($1 - \frac{1}{2}zz + \frac{1}{8}z^4$) gives $t = z + \frac{1}{3}z^3 + \frac{2}{15}z^5$ nearly.

P R O B.

375. *Any arc AM being given in degrees, to express that arc in parts of the radius.*

The circumference is to the diameter, or the semi-circumference to the radius, as (art. 244) 3.14159,2653589 is to unity: if, therefore, we say as the semi-circumference, or 180, is to the number of degrees of the arc, so is the number 3.14159, to that arc expressed in parts of the radius. Wherefore, if the arc be one minute, then as 180×60 , or 10800 minutes is to one minute, so is 3.14159, &c. to that arc 0.00029,08882, 086 expressed in parts of the radius.

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By

By means of the three last problems, and the following theorem, the sines, tangents, and secants, may be computed, as will appear by the following

E X A M P L E.

376. *To find the sine, cosine, and tangent of an arc of one minute.*

If the value of the arc z , found in the last problem, be substituted into (*art.* 372) $x = z - \frac{1}{2}z^2$, we shall find 0.00029,08882,086, for the sine; if the square of that value be substituted into (*art.* 373.) $v = 1 - \frac{1}{2}z^2$, we shall find 0.99999,99576,92 for the cosine; and lastly, $t = z$, gives 0.00029,08882,086 for the tangent required, true to the last figure.

T H E O R E M.

377. *If there be taken three arcs AN, AM, AL, in an arithmetical progression, the radius being unity; the sum of the sines of the extremes will be equal to twice the rectangle made by the sine of the mean and the cosine of half their difference; and the sum of their cosines to twice the rectangle made by the cosine of the mean and that of half their difference.* Fig. 2.

Draw the chord LN, to the difference between the least AN and the greatest AL; let the radius CM bisect at right angles the chord NL in R; then if the sines LD, MP, NQ, of these arcs are drawn, RI parallel to them, and ER, FN, parallel to the radius CA; the similar triangles CPM, CIR, gives (*art.* 67) $CM : CP :: CR : CI$, and $CM : PM :: CR : IR$. Therefore $CI = CP \times CR$, $IR = PM \times CR$, because the radius CM is unity.

Now because LR, RN, are equal, and ER, FN, parallel; LE and EF are also (*art.* 67) equal; so then DL, IR, and QN are in an arithmetical progression; that is, the sum of DL, QN, the extremes is equal to twice the mean IR: Therefore $DL + QN = 2PM \times CR$.

And since LE and EF are equal, ER or DI is half of FN or DQ; that is, DI and IQ are equal; and so $CD + CQ (=2CI) = 2CP \times CR$.

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378. If the arc MN or ML be one minute, its cosine CR will be (*art.* 376) 0.99999,99576,920: therefore $LD + NQ = PM \times 1.99999,99153,84$, and $CD + CQ = CP \times 1.99999,99153,84$.

By means of these two last equations, the sines and cosines of any arc from a minute to 30 degrees, may be found by a single multiplication.

E X A M P L E.

379. Let LD be the radius; then will PM be the sine of $89^\circ:59'$; that is, PM will be (*art.* 376) .99999,99576,92; and the cosine CP, 0.00029,08882,086; these values being substituted into the former equations, give $NQ = .99999,98665,067$ for the sine, and $CQ = .00058,18760,36$ for the cosine of the arc $89^\circ:58'$.

The next sine and cosine is found by supposing DL to be the sine of $89^\circ:59'$, and PM that of $89^\circ:58'$; and the operations may be thus continued to the finding of sines and cosines of arcs from a minute to 30 degrees.

380. If the arc AM be 30 degrees, its sine PM will be equal to half the radius; or, because the radius is unity, that sine will be $\frac{1}{2}$: therefore the equation (*art.* 377) $LD + NQ = 2PM \times CR$, will become $LD + NQ = CR$ in this case. *Which shews that the sum of the sines of two equidistant arcs from 30 degrees, is always equal to the cosine of half their difference.*

And because of the similar triangles CPM, LFN, we have $CM (1) : PM (\frac{1}{2}) :: LN : FN$ or $:: 2LR : DQ$: Therefore $LR = (DQ) CQ - CD$. Consequently, *the difference between the cosines of any two equidistant arcs from 30 degrees, is always equal to the sine of half their difference.*

By means of these equations the sines and cosines of any arc between 30 and 60 degrees, may be found by subtraction only, when the sines and cosines of all the arcs from a minute to 30 degrees are known.

The tangents may be found much in the same manner as the sines and cosines, by the expression given for

that purpose; or because the cosine CP (Fig. 1.) is to the sine PM, as the radius CA is to the tangent; the sine and cosine of an arc being given, the tangent is easily found by this last proportion.

The cosine CP is also to the radius CM as the radius CA to the secant CT. Consequently, the cosine of an arc being given, the secant may be found.

EXPLANATION of the TABLES.

At the top of the page is placed the number of degrees; in the first vertical column to the left, the minutes; in the second vertical column the sines; in the third the tangents; in the fourth the secants; and then follow the logarithmical sines and tangents; and in the opposite page are the sines, tangents, and secants of their complements, in the same order as in the first.

Thus, if the sine, tangent, and cosine of $39^{\circ} 15'$ is required, look for 39 degrees at the top of the page, and for the 15 minutes in the first vertical column to the left; then the number 63270.53 against 15, and under the word sine, will be the sine; the number 81703.43, against 15, and under the word tangent is the tangent; and in the opposite page the number 77439.26, in the same horizontal line under the word sine, is the cosine of the arc required.

Again, the sine of 34 degrees 40 minutes, is 56880.11, the tangent 69157.24, the secant 121584.23, and cosine 82247.51.

If the sine or cosine of any angle greater than 90 degrees is required, subtract from 180 the given number, and the sine or cosine of the difference will be that required: for it has been shewn, that the sine or cosine of any arc under 90 degrees, was also the sine or cosine of its supplement. Thus, if the sine of 100 degrees is wanted, subtract 100 from 180, and the sine 98480.77 of the difference will be that required.

As the tables are computed to minutes only, and it is sometimes required to find them to a greater accuracy, we shall shew how to find them nearer by approximation,

mation, or from the given sine or tangent to find the angles corresponding thereto to a great exactness. Fig. 3.

381. Let EFG be such a curve, that if the ordinates BE, CF, DG, to the axis AD, express any sines or tangents, the distances AB, AC, AD, the corresponding angles to these sines or tangents; from the extremity E of the least ordinate draw a line parallel to the axis AD, meeting CF in I, and DG in H; then if the interval BD between the extremes is not above a minute, the differences between the sines will be proportional to the differences between the angles nearly.

That is $EH:HG::EI:IF$, nearly; and if FL be drawn parallel to EH, there will also be $EH:HG::FL:LG$, nearly.

Example I. Let the angle AC be $30^{\circ}:50':50''$; then if AB be $30^{\circ}:50'$, and AD $30^{\circ}:51'$; because the sine BE is 5125425, the sine DG, 5127922, and their difference HG, 2497; BD or EH will be one minute or 60 seconds, and CD or FL, 10; therefore $EH(60):HG(2497)::FL(10):LG(416)$; this being subtracted from DG, gives 5127506, for the sine CF required.

It must be observed, that if the ordinate CF is nearer DG than BE, the difference LG is to be found and subtracted from the greater DG, in order to get CF; but if CF is nearer to BE than DG, the difference IF is to be found and added to the less BE.

Example II. Let 4567890 be the sine whose corresponding angle is required; as this sine is greater than 4565804, that of $27^{\circ}:10'$; and less than 4568392, that of $27^{\circ}:11'$; if we suppose BE to be 4565804, DG, 4568392, then will GH be 2588; LG, 502; and therefore $HG(2588):EH(60)::LG(502):FL(11.64)$; this being subtracted from EH (11'), gives $10':48.36''$. Consequently, AC, $27^{\circ}:10':48.30''$, will be the angle required.

The same rule is to be observed for the tangents or their logarithms. Having thus shewn the use and construction

struction of the tables, we shall now proceed to the solution of triangles.

SOLUTION of RIGHT ANGLED TRIANGLES.

T H E O R E M.

382. Fig. 4. If in any right angled triangle CDE, from one of the acute angles C as center, the arc AM be described with any radius, and the tangent AT to that arc be drawn, as likewise the sine PM; then by reason of the similarity of triangles (art. 69) we have $CP:PM::CA:AT::CD:DE$, and $CP:CM::CA:CT::CD:CE$. Which shews, that the cosine is to the sine, or the radius is to the tangent, as the adjacent side CD to the angle, is to the opposite side DE.

And the cosine is to the radius, or the radius is to the secant, as the adjacent side CD is to the hypotenuse CE.

C A S E I.

383. Fig. 5. Let the sides CD, DE, of a triangle right angled at D be given, to find the acute angles at C and E.

If $CD=100$, and $DE=60$, then as CD (100) is to DE (60) so is the radius 100000 to the tangent 60000 of the angle C: now looking into the tables of the sines and tangents for the angle corresponding to that tangent 60000, we shall find $30^{\circ}:58'$, nearly. Therefore the adjacent angle C is $30^{\circ}:58'$, and the angle E, the complement to the angle C is $59^{\circ}:2'$.

C A S E II.

384. Let one of the sides CD and an acute angle C, be given, to find the other side DE.

If the side $CD=100$, and the angle C 40 degrees; then the radius 100000 is to the tangent 83909 of 40 degrees, as the side CD (100) is to the side DE (83.909) required.

C A S E III.

385. Let one of the acute angles C and one side CD be given, to find the hypotenuse CE.

If

If the angle C be 40 degrees, and the side CD = 100; then the radius 100000 is to the secant 130540 of 40 degrees, or of the angle C, as the side CD (100) is to the hypotenuse CE (130.54) required.

These are all the cases that can happen in right angled triangles; for when one of the acute angles is given, the other, which is its complement, will also be given; and when the two sides are given, the sum of their squares is (art. 47) equal to the square of the hypotenuse.

T H E O R E M.

386. In any triangle ABC, the sides are to each other as the sines of the opposite angles. Fig. 6, 7.

If from any of the angles B, there be drawn a perpendicular BE to the opposite side AC; then if the radius be called R, and the sines of the angles A, C; $\sin A$, $\sin C$,

We have $\{\sin A : R :: BE : AB\}$ Therefore (art. 85) $\sin A : \sin C :: BC : AB$
 $\{\sin C : R :: BE : BC\}$

T H E O R E M.

387. In any triangle where a perpendicular CD is drawn to the base, the base AB is to the sum of the sides AC, BC, as their difference is to the difference between the segments AD, DB. Fig. 8.

This has been proved in art. 50.

T H E O R E M.

388. In any triangle ABC, the sum of the sides AC, BC, is to their difference as the tangent of half the sum of the opposite angles A, B, is to the tangent of half their difference. Fig. 9.

Let there be the same construction as before, draw AE, AF, and FG parallel to EA; then as the external angle ECA is equal to the two internal (art. 8) opposite ones A, B; and the angle EFA, which is at the circumference, is (art. 109) half the angle ECA at the center; the angle EFA is half the sum of the angles A and B; and since the angles CAF, CFA, in the isosceles triangle ACF, are equal, the angle FAG, which is

the difference between the greater angle CAB and half the sum CAF or CFA, will be half their difference.

Now because the angle FAE is contained in a semi-circle, it will be a (*art.* 114) right angle; and since FG is parallel to AE by construction, the angle AFG will also be a right angle: if therefore AF be taken for the radius, AE will be the tangent of AFE, half of the sum, and FG the tangent of half the difference FAG, of the angles A and B; and by reason of the parallels AE, GF, we have (*art.* 67) $BE:BF$, or $BC+AC:BC-AC::AE:GF$.

SOLUTION of OBLIQUE TRIANGLES.

C A S E I.

389. *Two sides of any triangle ABC being given, together with an angle opposite to one of these sides, to find the rest of the angles and side.* Fig. 10.

If $AC=100$, $AB=60$, and the angle B 100 degrees, then as 80 degrees is the supplement of 100, if we say (*art.* 386) as AC (100) is to AB (60) so is the sine 98480 of 80° , to the sine 59088 of the angle C, which therefore is $36^\circ:13'$; and subtracting the sum $136^\circ:13'$, of the two known angles B, C, from 180, we shall have (*art.* 12) $43^\circ:47'$ for the angle A.

Whence the sine 98480 of the angle B, is to the sine 69193 of the angle A, as the side AC (100) is to the side $BC=70.26$.

N. B. If the acute angle A is given, and the sides AB, BC, then if the side BC is less than BA, the angle C opposite to the side AB, may either be acute or obtuse; for there may be drawn another line equal to BC, such that the angle it makes with AC will be the supplement to the angle BCA; and therefore it must be known whether the angle sought is acute or obtuse before this case can be solved.

C A S E II.

390. *Two angles and a side of any triangle being given, to find the other sides.*

If the angle A is 25 degrees, the angle B, 80, and the side AB, 120; subtract the sum 105 of the two given

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given angles A, B, from 180 degrees, then the difference 75 will be the value of the angle C; therefore the line 96592, of the angles C (75) is to the line 98480 of the angle B (80) as (art. 386) the side AB (120) is to the side $AC = 122.34$.

And the line 96592 of the angle C (75) is to the line 42261 of the angle A (25) as the side AB (120) is to the side $BC = 52.5$.

C A S E III.

391. *The three sides of any triangle being given, to find the angles.*

If the side $AC = 120$, $BC = 80$, and $AB = 100$; then the (art. 387) base AC (120) is to the sum 180 of the sides AB, BC, as their difference 20 is to the difference between the segments AD, DC, which is 30. This being added to the base 120; then half the sum which is 75, will be the value of the greater segment AD.

Now in the right angled triangle ADB, having the sides AD and AB, the angle A is found by saying, (art. 382) as AD (75) is to AB (100) so is the radius 100000, to the secant 133333 of the angle A, which will be 41 degrees and 25 minutes.

And in the triangle ABC, the sides AB, BC, being given, the angle C will be found by saying as BC (80) is to AB (100) so is the sine of the angle A (66153) to the line 82691 of the angle C, which therefore is 55 degrees and 47 minutes; and the difference $82^\circ : 48'$, between the sum $97^\circ : 12'$ of the angles A, C, and 180 degrees will be the value of the angle B.

C A S E IV.

392. *Two sides with the including angle of any triangle being given, to find the other angles and side.*

Let $AC = 100$, $AB = 60$, and the angle A, $53^\circ : 48'$; then as (art. 388) the sum 160 of the sides AB, AC is to their difference 40, so is the tangent 197110 of $63^\circ : 6'$ half the sum of the opposite angles B, C, is to the tangent 49277 of half their difference; this answers to an angle of $23^\circ : 54'$.

Now if we add this angle $23^\circ : 54'$, to half the sum $63^\circ : 6'$, we shall have 87° , for the greatest angle B; and

and subtracting $23^{\circ} : 54'$ from $63^{\circ} : 6'$, we shall get $39^{\circ} : 12'$, for the least angle C.

Hence the sine 99862 of the angle B (87°) is to the sine 80696 of the angle A ($53^{\circ} : 48'$) so is the side AB (100) to the side $BC = 80.807$.

These are all the different cases that can happen in any plane triangle; and to shew their use we shall add the following problems, which most commonly occur in practice.

P R O B L E M.

393. *To find the distance from an inaccessible object A, placed beyond a river, or precipice, to a given point B.*
Fig. 12.

On a level spot of ground measure the base BC, place the instrument at B, and take the angle CBA; carry the instrument to the other extremity C of the base, and take also the angle ACB: Then having the base BC of the triangle ABC, and the adjacent angles B and C, the distances AB, AC, may be found by case I.

If the angle B be 70 degrees, the angle C, 68, and the base BC, 120 yards, by subtracting the sum 138 of the angles B and C from 180, the difference 42 will be the value of the angle A. Therefore, the sine 66913 of the angle A (42) is to the sine 92718 of the angle C (68) as the base BC (120) is to the distance $AB = 166.27$ yards.

If the triangle ABC be laid down by a scale of equal parts and a protractor, and the sides AB, AC, being carried on the scale with your compasses, they will be found pretty near the same as by computation.

P R O B L E M.

394. *To find the distance between any two inaccessible objects A and D.*

In a convenient place measure the base BC, as likewise the angles made by that base, and the lines drawn from its extremities to these objects; then in the triangle ABC, having the base and the adjacent angles, the side AC may be found by case I. And in the triangle BCD, having the base BC with the adjacent

cent angles, the side CD may be known. Lastly, in the triangle CAD, having the sides CA, CD, with the included angle, the inaccessible distance AD may be found by case IV.

If the base $BC = 125$ yards, the angles ACD (50°), ACB (42°), ABC (86°), CBD (40°); then the difference between the sum 128 of the angles ABC , ACB and 180 gives 52 for the angle BAC ; and therefore, the sine 78801 of the angle A (52°) is to the sine 99756 of the angle B (86°) as the base BC (125) is to the side $CA = 158.24$.

The sum 132 of the angles BCD , DBC , being subtracted from 180 gives 48 for the angle CDB ; whence, the sine 74314 of the angle D (48°) is to the sine 64278 of the angle B (40°) as the base BC (125) is to the side $CD = 108$.

Lastly, the sum 266.24 of the sides CA , CD , is to their difference 50.24 as the tangent 214450 of 65 half the sum of the angles A , D , is to the tangent 40310 of half their difference, which is $21^\circ : 58'$; this being added to and subtracted from half the sum 65 , gives $86^\circ : 58'$, for the greatest angle CDA , and $43^\circ : 2'$, for the least CAD . Consequently, the sine 68242 of the angle CAD ($43^\circ : 2'$) is to the sine 76604 of the angle ACD (50°) as the side CD (108) is to the distance required $AD = 121$.

P R O B L E M.

395. *To find the height of an accessible object AB. Fig. 13.*

Place the instrument at a convenient distance from the object as at C ; let CD be the height of the Instrument, and observe the angle BDE made by the horizontal line DE and the line DB ; measure the distance CA or DE : then as all the angles in the right angled triangle DEB are known, as also the base DE ; the perpendicular EB (*art. 384.*) may be found, to which adding the height AE of the point E from the ground, and you will have the height required.

Let the base CA or DE be 100 feet, and the angle EDB , 60 degrees; then as the radius 100000 is to the tangent 173205 of the angle D (60°) so is the side DE (100) to the perpendicular EB (173.205) to which adding

ding 4 feet for the height of the point E from the ground you will have $AB = 177.205$.

P R O B L E M.

396. *To find the height of an inaccessible object AB.*

Place the instrument in a convenient place as at C, and observe the angle BDE, made by the horizontal line DE and the line DB; then go backwards in a direct line with the points D and E, at a convenient distance as at F, and take the angle BFE and measure the distance FD: then having all the angles of the triangle FDB and the base FD, the side DB may (*art.* 390) be found; and having the side DB and the angle BDE, the height EB may (*art.* 384) be found.

If the angles BDE ($62^\circ : 50'$), BFD ($48^\circ : 30'$) and the base DF, 100 feet, then because the exterior angle BDE is equal to the sum of the two interior opposite ones F and B; the difference $14^\circ : 20'$, between the angles BDE, BFE, will be the angle FBD. Therefore, the sine 24756 of the angle B ($14^\circ : 20'$) is to the sine 74895 of the angle F ($48^\circ : 30'$) as the base FD (100) is to the side $DB = 302.53$; and the radius 100000 is to the sine 88968 of the angle BDE ($62^\circ : 50'$) as the side DB (302.53) is to the perpendicular $EB = 269.15$; to which adding 4 feet for the height of the point E from the ground, you will have $AB = 273.15$ for the height wanted.

P R O B L E M.

397. *The distance between three objects A, F, B, lying in a right line, being given together with the angles under which they are seen from any place D, to find the distances from this place to the three objects.*

Suppose a circle to pass through the two extreme objects A, B, and the point D; let a line be drawn from the point D and the third object F, meeting the circumference in E, draw AE, BE; then because the angles EAB and EDB, insisting on the same arc EB, are (*art.* 113) equal, as well as the angles EBA and EDA, we have the three angles of the triangle AEB and the side AB, and so the side AE may be found; and having

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ing the sides AE, AF, with the included angle, the angle AFE may be found. Lastly, in the triangles ADF, BDF, having the angles at D and F, and the sides AF, FB, the required distances may be known.

If $AB=180$ fathoms, $AF=50$, the angle ADF, $21^{\circ}:15'$, and the angle BDF, 38 ; then will the angle AEB be $120^{\circ}:45'$; whence, the sine 85940 of the supplement $59^{\circ}:15'$ to the angle AEB, will be to the sine 36243 of the angle ABE, or of its equal ADE ($21^{\circ}:15'$) as the side AB (180) is to the side AE = 75.9 or 76 fathoms. Now the sum of 126 of the sides AE, AF, is to their difference 26 as the tangent 290421 of 71 half the sum of the angles E, F, is to the tangent 59928 of half their difference, which answers to an angle of $30^{\circ}:56'$, this added to 71 , gives $101^{\circ}:56'$, for the greatest angle AFE or BFD.

Therefore, the sine 61566 of the angle FDB (38) is to the sine 97838 of the supplement $78:4'$, to the angle DFB ($101^{\circ}:56'$) so is the side FB (130) to the side DB = 206.5.

And the sine 61566 of the angle FDB (38) is to the sine 64367 of the angle DBF ($40^{\circ}:41'$) so is the side FB (130) to the side FD = 136.

Lastly, the sine 36243 of the angle ADF ($21^{\circ}:15'$) is to the sine 97838, of the angle AFD ($78^{\circ}:4'$) so is the side AF (50) to the distance AD = 135.

This problem is very useful in the attack of a place; for by means of Dr. *Hadley's* reflecting quadrant, the points of the bastions before the attack may be seen as well as one of the shoulders, by looking between two sand bags; the distances from one point of the bastion to that of the next, as well as to the shoulder, are generally known from the author's method that fortified the place; and from thence, the distance from the trenches to the place are easily known; for though these three points are not exactly in a line, yet they are near enough to cause no great error.

If the point D is chosen in the capital of the ravelin produced, the line DF will be perpendicular to the exterior

terior side AB; then the problem will be much easier; since the radius will be to the secant of the angle DAB or DBA, as half the exterior side AB is to the side AD or BD.

P R O B L E M.

398. *The distances between three objects A, B, C, not placed in a right line being given, together with the angles under which they are seen from any place D; to find the distances from that place to the three objects.* Fig. 11.

Suppose a circle to pass through any two of these objects A, B, and the point D, and let the line CD intersect or meet the circumference in E, draw AE and BE; then as the angles EAB, EDB, insist on the same arc EB, they (*art. 113*) are equal, as likewise the angles EBA and EDA; whence, in the triangle AEB, the side AB, and the adjacent angles are given, as being equal to the angles at D, therefore the side AE is also given: and because the three sides of the triangle ABC are given by supposition, the angles are (*art. 391.*) likewise given; and so the angle CAE, which is the sum or difference of two given angles CAB and BAE, is also given; then as the sides AC, AE, and the included angle are given, the angle ACE may (*art. 392*) be found.

Lastly, having the side AC and all the angles of the triangle DAC given, the sides DA and DC may (*art. 390*) be found; and having in the triangle DBA, the sides AB, AD, and the angles at A and D, the side BD may be (*art. 389*) found.

If $AB=100$, $AC=66$, $BC=70$, and the angles ADC, 27, CDB, 32, then as the sine 85716 of the supplement 59 to the angle AEB is to the sine 45399 of the angle EBA (27) so is the side AB (100) to the side $AE=53$: to find the angle CAB say, as the base AB (100) is to the sum 136 of the sides AC, BC, so is their difference 4 to the difference 5.44, between (*art. 384*) the segments of the base, this being subtracted from the base 100; half the difference 47.28 will be the segment next to the angle A; but this segment 47.28 is (*art. 386*) to AC (66) as the radius 100000 is

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to the secant 13959 of the angle A, which therefore is $44^{\circ}:14'$, and hence the angle CAE is $12^{\circ}:14'$, as being the difference between the angles BAC, ($44^{\circ}:14'$) and BAE, or its equal BDC (32). Now as the sum 119 of the sides AE, AC, is to their difference 13, so is the tangent 933154 of $84^{\circ}:53'$, half the sum of the angles C, E, to the tangent 101941 of half their difference; which answers to an angle of $45^{\circ}:33'$; this being subtracted from half the sum $83^{\circ}:53'$, gives $38^{\circ}:20'$ for the least angle ACD.

Therefore, the sine 45399 of the angle ADC (27) is to the sine 62023 of the angle ACD ($38^{\circ}:20'$), as the side AC (66) to the side AD=90.16. And as the sine 45399 of the angle ADC (27) is to the sine 90875 of the supplement $65^{\circ}:20'$ to the angle DAC, so is the side AC (66) to the side DC=132.2. Lastly, in the triangle DAB, the sine 85716 of the angle ADB (59) is to the sine 94225 of the angle DAB ($70^{\circ}:26'$), as the side AB (100) to the side DB=110.

If the point C falls any where in the circumference, the problem is indetermined; for the angles CAB, CBA, are in this case equal to the alternate angles at D; and so any point in the arc ADB will satisfy the problem.

This last problem is useful in surveying the plan or map of a country; for having once the distances between three towns, those to any other from which these are seen may be found; and in this manner the distances between all the towns may be found one after another.

TRIGONOMETRY *applied to* FORTIFICATION.

When a plan of a fortification is to be traced on the ground, the masonry to be computed, or an estimate is to be made, it is necessary to know exactly all the lines and angles which compose it; for which reason we shall give the following specimen, applied to a front constructed according to Mr. *de Vauban's* first method, which being well understood, may easily be applied to any other construction whatsoever.

P R O B.

399. *To find the lines and angles of the front of a fortification, constructed according to Mr. de Vauban's first method.* Plate XIV. Fig. 9

The exterior side AB is supposed to be 180 fathoms or toises; the perpendicular CD, 30; the faces AE, BH, 50; and the lines EH, EG, and FH, are equal; we are from these data to determine the rest of the lines and angles.

The first thing to be found is the angle CAD, made by the exterior side and the faces of the bastions: As in the right angled triangle ACD, we have the sides AC (90) and CD (30) the angle A will be found to be 18 degrees and 26 minutes; twice this angle, subtracted from the angle of the polygon, gives the flanked angle of the bastion: In an exagon, the angle of the polygon is 120 degrees; and therefore the flanked angle is 83 degrees and 8 minutes.

The side AD is to be found next, which is 94.87 fathoms; and subtracting the face AE (50) from AD, we shall have 44.87 for the line ED; whence, the similar triangles ACD, EPD, give 42.57 fathoms for EP; and therefore EH and EG, which are double of EP, will each be 85.14 fathoms.

Now in the isosceles triangle EHF, having the angle EHF equal to the angle CBD ($18^{\circ} : 26'$) the angles at the base EF will each be 80 degrees and 47 minutes; whence, having the two sides EH, FH, and all the angles, the flank EF will be found to be 27.27 fathoms. But in the triangle GFE, having the sides EF, EG, the angle FGE, equal to its alternate one DAC, ($18^{\circ} : 26'$) and the angle GFE at the curtain is equal to the sum of the angles HFE ($80^{\circ} : 47'$) and GFH ($18^{\circ} : 26'$) and so is 99 degrees and 13 minutes, and the angle FEG 62 degrees 21 minutes; by which the length of the curtain FG is found to be 76.39 fathoms; and subtracting the angle FEG of 62 degrees and 21 minutes from two right angles, we shall have 117 degrees 39 minutes for the angle AEF of the shoulder.

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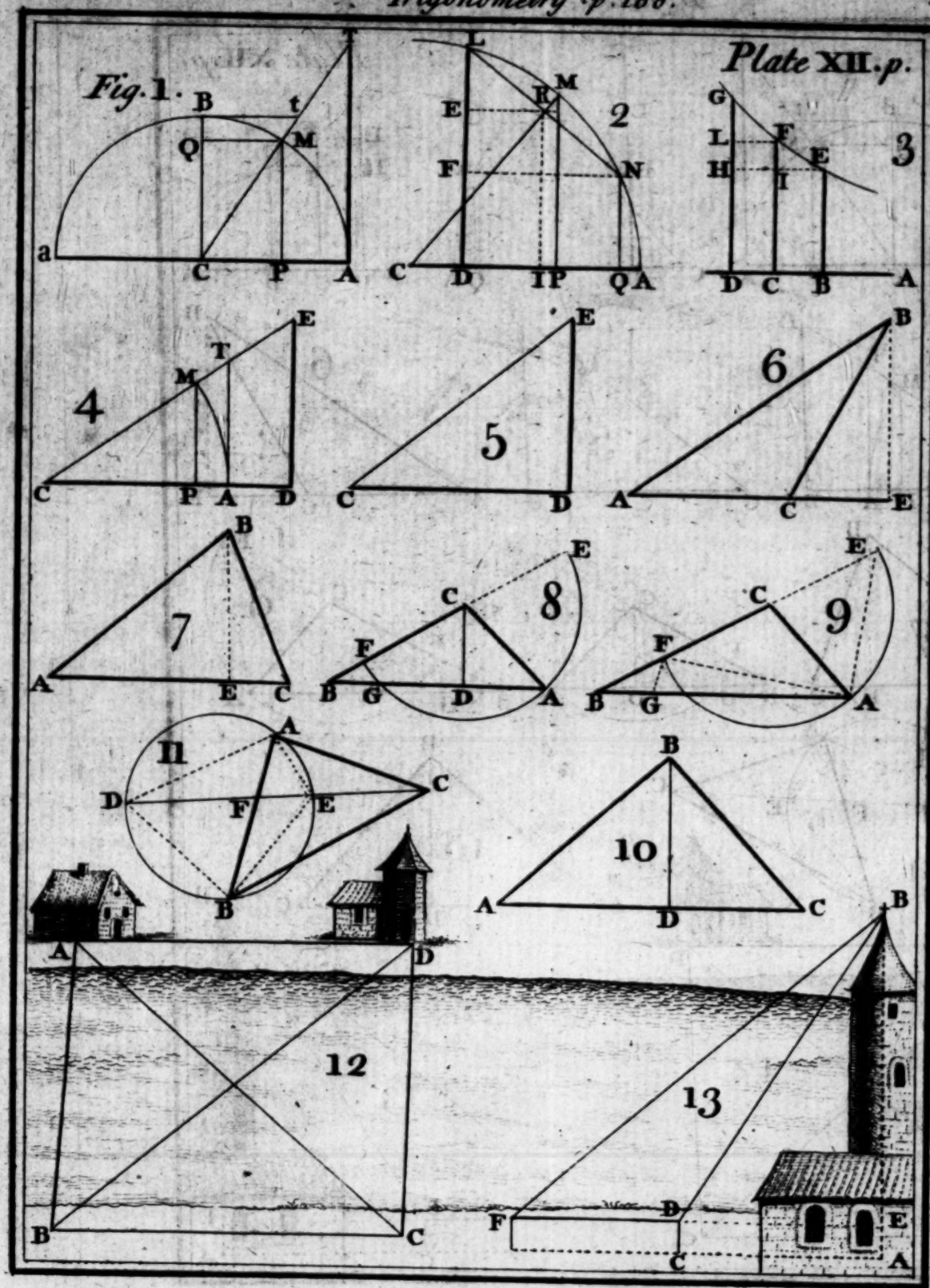
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These are all the lines and angles which compose the body of a place: but as they are often made with orillons and retired flanks, we shall shew how they are to be found.

PROBLEM.

400. *To find the length of the arc RS, which forms the retired flank.*

This flank is constructed in the following manner; the strait flank GH is found as usual, and divided into three equal parts, so that GT is two of them, and TH one; from the opposite flanked angle A, a line AT is drawn, in which the part TS is taken equal to 5 fathoms; and in the line of defence AG, produced, the part GR is taken equal to TS; then on the base RS is described an equilateral triangle Rps, and from the opposite angle p, as center, the retired flank is described.

As the face AE is 50 fathoms by construction, and EG has been (*art.* 399) found to be 85.13, the side AG will be 135.13 fathoms; now in the triangle AGT, having the sides AG, GT, with the included angle AGT, of 80 degrees 47 minutes, the angle ATG will be found (*art.* 392) to be 91 degrees 29 minutes, and the angle GAT 7 degrees 44 minutes; and from thence the side AT is found to be 133.36 fathoms.

If to AT we add TS, or 5 fathoms, we get 138.36 for AS; and if to AG (135.13) we add GR (5) we have AR equal to 140.13 fathoms. Now in the triangle RAS, having the sides AR, AS, with the included angle A of 7 degrees 44 minutes, the angle ARS at the base will be found (*art.* 392) to be 80 degrees 45 minutes; and having the side AS, and the angles of the triangle ARS, the side RS is found to be 18.86 fathoms.

As the radius Rp is equal to the chord RS by construction; if we say as (*art.* 352) unity is to 3.1416, so is the radius pR (18.86) to its semi-circumference, we shall have 59.25 for the semi-circumference of the radius pR: And since the arc RS is 60 degrees, or one

third of two right angles, the retired flank RS will be 19.75 fathoms.

PROBLEM.

401. *To find the orillon TH, according to Mr. de Vau-
ban's construction.*

If from the extremity H of the face BH of the bastion, a perpendicular Hn is drawn, and another which bisects TH at right angles; their intersection n is the center from which the orillon is described.

If from the angle of the shoulder (*art.* 399) BHG ($117^{\circ}:39'$) we subtract the right angle BHn, we get $27^{\circ}:39'$, for the angle nHr; and as TH is 9.09 one third of the flank GH, half of this gives 4.545 for Hr: Therefore, in the right angled triangle Hrn, having one side Hr and all the angles, the radius Hn of the orillon will be found to be 5.13 fathoms. Now if we say as unity is to 3.1416 so is the radius Hn (5.13) to its semi-circumference 16.116 fathoms; and as the angle Hnr is $60^{\circ}:21'$, twice that angle $120^{\circ}:42'$ or 120.7 will express the orillon in degrees.

If therefore we say as 180 degrees is to 16.116 fathoms, so is 120.7 degrees to the orillon 10.8 fathoms.

PROBLEM.

402. *To find the length of the counterscarp ILR.*

The arc LR or round part of the counterscarp before the flanked angle of the bastion, is described with a radius of 20 fathoms; and the counterscarp produced meets the shoulder H. If AL be drawn perpendicular to IL, and joining the line AH, we have in the right angled triangle ALH the side AL and the right angle, which is not sufficient to determine the rest; for which reason the side AH must be found by some other triangle.

In the triangle AEH, having AE 50 by construction, and EH equal (*art.* 399) to 85.14, with the included angle AEH, the supplement to GEH of 18 degrees 26 minutes, we shall find the angle AHE to be 6 degrees 48 minutes; and from thence the side AH comes out to be 133.53 fathoms. Now, in the right angled

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angled triangle ALH, having the sides AH and AL, the angle AHL will be found to be 8 degrees 37 minutes; and its complement LAH, 81 degrees 23 minutes. Hence the side LH is 132 fathoms.

In order to get the length IL of the counterscarp, the part IH must be found, which is done by considering that in the right angled triangle HPI, the angle EHA has been found to be 6 degrees 48 minutes, and the angle AHL 8 degrees 37 minutes; so that the angle PHI is 15 degrees 25 minutes; and since the side PH is 42.56, the side IH will be found to be 44.15 fathoms; and this subtracted from LH (132) gives 87.85 fathoms for the strait part IL of the counterscarp.

To find the arc LR, or half the circular part before the bastion, it is necessary to know the external angle RAC, which in an exagon is 120 degrees; and as the angle HAC is equal to the alternate angle AHE, 6 degrees 48 minutes; by adding this angle to the angle CAR (120) we shall have 126 degrees 48 minutes for the angle HAR, from which subtracting the angle HAL, 81 degrees 23 minutes, we shall have 45 degrees 25 minutes for the angle LAR; or because $\frac{25}{60}$ is nearly equal to .4171 decimals of a degree, the arc LR will be 45.417 degrees.

Now if we say as unity is to 3.1416 so is the radius AL (20) to its semi-circumference, 62.832; and then as 180 degrees is to 62.832, so is 45.417 degrees to the arc LR = 15.853 fathoms.

P R O B L E M.

403. *To find the several parts of the ravelin NQM.*

We suppose the capital IQ to be 50 fathoms, and the faces produced to meet those of the bastions, in a point O within three fathoms from the shoulder E. Hence, if we find DI we shall have in the triangle DQO the sides DQ and DO, with the included angle, by which the rest is determined.

As the angle PHI was found to (art. 402) be 15 degrees 25 minutes, and the side PH (art. 399) 42.56 fathoms, the side PI will be 11.74 fathoms; and by

the similarity of the triangles ACD , EPD , we get $PD = 14.19$ fathoms; this being added to PI (11.74) gives 25.93 fathoms for DI ; and if to DI we add the capital IQ (50) we get 75.93 fathoms for the side DQ . The side DE has been found (*art.* 399) to be 44.87 fathoms, to which adding EO (3) gives 47.87 for DO : and since the angle ADC is the complement to the angle DAC (*art.* 399) 18 degrees 26 minutes, that angle will be 71 degrees 34 minutes, the angle DQO will be found to be 36 degrees 45 minutes; and therefore the salient angle NQM of the ravelin is 73 degrees.

The angle PHI was found (*art.* 402) to be 15 degrees 25 minutes, its complement PIH , or NIQ , will be 74 degrees 35 minutes; and therefore, in the triangle INQ , having the side IQ (50) by construction and the adjacent angles, the side QN or the face of the ravelin will be 51.75 fathoms; the semigorge IN , 32.12; and the angle INQ , 68 degrees 40 minutes.

It remains now to find the counterscarp of the ditch before the ravelin, in order to which draw QT , QV , perpendicular to these counterscarps; then as they are parallel to the faces of the ravelin, the lines QT , QV , are also perpendicular to the faces; and since the four angles at Q are (*art.* 4) equal to four right angles, and the angles NQV , MQT , being right ones, the angle TQV will be 107 degrees, the complement to the angle NQM , 73 degrees. Whence, if we say as unity is to 3.1416, so is the radius $Q\Gamma$ (12) to the semi-circumference 37.6992; and again, as 180 degrees is to 37.6992; so is 107 degrees to the arc $TV = 22.46$ fathoms.

If Nb be drawn perpendicular to the counterscarp aV , the part bV will be equal to the face NQ (51.75 fathoms) and the angle Nab will be 68 degrees 40 minutes equal to the angle INQ , its complement aNb , 21 degrees 20 minutes. Whence the side $Na = 12.88$, and $ab = 4.36$ fathoms.

S E C T.

S E C T. II. Of S U R V E Y I N G.

OF all the instruments invented for the use of surveying, there is only the Plain Table, and the new improved Theodolite, that are worth mentioning, the rest being more for show than for any real use; for which reason we shall describe the manner of using these two only.

The plain table is a rectangular board of about 20 inches long, from 15 to 18 broad, and about half an inch thick; it is made of fir, mahogany, or any other dry wood which does not warpe; about this board is a frame of about an inch broad, let into it, in order to fasten the paper; this instrument is supported like all others, by a three legged staff, having a ball and socket.

There is also a long brass ruler with sights, so placed, as to answer one of the edges which serves to find the directions of objects. There is likewise a needle and compass fixed to it, and the degrees of a circle are marked on the frame of some; but these things are of no real use, for which reason I never have them made.

For, if at 12 o'clock a thin wire or pin is stuck upright on the paper, its shadow gives the meridian line of the ground much better than the needle.

P R O B L E M.

404. *To survey a spot of ground with a plain table.*

Before you begin it will be proper, tho' not absolutely necessary, to go round the ground with some person who is acquainted with its boundaries, in order to observe the most convenient place for the stations, which should always be as near to them as possible, sometimes within and sometimes without, according as the nature of the ground will permit it; those are to be pitched upon where one can see farthest, forwards and backwards, and where the chain may be carried in a direct line from one to another.

This being premised, place the table at the second station B, lay the plane of it as nearly level as you can, place also two station staves, the one at A and the other

at C; chuse a point on the paper to represent the place B; turn the brass ruler about it till you see, through the sight, the station A, and draw a pencil line along the edge; then turn the ruler still about the point B, till you see the station C, and draw another line along the edge of the ruler; these two lines will make the same angle on the paper as the three places A, B, C, make with each other on the ground.

This done, two men with a chain measure the distances from B to C, and from B to A; the foremost takes a dozen of arrows, or sticks, and fixes one at every chain's length in the ground, which he that follows, holds the end of the chain near, till it is stretched, and then takes it up; and they continue thus till the fore part of the chain reaches the station; if it exceeds it by some links they must be taken notice of; and the number of arrows taken up by him who follows the chain, will give the number of chains measured between the stations.

Care must be taken that the chain is carried in a strait line, and as nearly horizontal as possible; he who follows the chain must direct him that precedes; and, on the contrary, the first by looking backwards will be able to direct himself: It is necessary to have these distances very exact, because the exactness of the survey depends chiefly thereon.

In the same time that the distances from one station to another are measured, the offsets at every chain's end are also measured, or sometimes at a less or greater distance, according as the boundary is more or less crooked, and marked in a pocket-book, together with the turnings of hedges, lanes, houses, stiles and gates, and the names of the adjoining fields; for all these things must be represented in the plan.

Suppose that the distance AB has been laid down on paper, draw perpendiculars to AB at every chain's length, on which set off the several offsets noted in the book, and through their extremities the boundaries are drawn so as to represent a hedge, ditch, lane, or whatever else is on the ground.

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This being done the station-staff at A is carried to B, and some mark left there to find the place again, if required; the instrument is removed to the station C, and another station at D; the table being laid level, or nearly so, it must be turned so as the line CB on the paper be exactly in the direction of that line on the ground; then the brass ruler is turned round the point C, till the fourth station D is seen through the sights; draw the line CD; then the line CD being measured with the same care and exactness we mentioned before, as likewise all the offsets, and notice taken of every remarkable object; and when these things are marked on the paper, the instrument is to be removed to the fourth station D, the staff to C, and another to E.

The instrument being laid level, and turned so as the last line CD on the paper be in the direction of that line on the ground, the brass ruler is turned about the last point D, till the next station E is seen through the sights; then draw the line DE; proceed in the same manner at every station, till you come to the last, where you direct the line to the first station A, and if the distance between the first and last station answers exactly on the paper, you are certain that the survey is rightly performed; but if it does not, some errors have been committed, either in taking the angles, or in measuring the distances.

If the ground be divided into small fields, and there are any buildings, ponds, or rivers, in such a case it is best to take the boundary of the whole at first, if it can be brought into one sheet of paper; which being found exactly and laid down, the internal parts are surveyed afterwards, by setting out from some of the former stations: But if the extent of the ground is considerably large, so as not to be taken all on one sheet, it will be more convenient to take the fields and other objects one after another, till the whole is finished.

If there are but a few objects within the limits of the survey, such as trees, buildings, and gardens, in the manner marked at P; then from any station as G, where the corner of the house may be seen, you direct

the ruler to it, and draw a line; and if you do the same thing from some other stations as H, I, K, the intersections of these lines will give the situation of the building on the plan.

If there be a river, such as is marked here; from the next station D you direct the ruler to a station L; the distance from D to L being set off, you proceed thence along the river from L to M, and from M to N, quite to the end of the boundary; and as you go along take the offsets, especially where the river bends most, and here and there the breadth, by which the course of the river will be determined.

Of the THEODOLITE.

THIS instrument is composed of a brass circle 9 or 10 inches diameter, divided into degrees, called the Limb, and of a brass ruler which turns about the center, having one end divided, according to *Nonius*, called the Index; and upon this index is fixed a vertical semi circle divided into degrees, with a telescope sliding up and down, having likewise an index under it divided according to *Nonius*. There is also fixed a box and needle to the index, with two spirit-levels at right angles to each other, to lay the plane of the instrument truly horizontal: The instrument is supported by a three-legged staff in the same manner as the plain table; and instead of the ball and socket, four screws, playing between two brass plates, are used, which keep it much steadier and firmer.

In order to survey with a theodolite, it is necessary to have a field-book divided into three columns; the middle one serves to mark the angles and distances from one station to another, and the other two to insert the offsets, either on the right or left, according as they are situated; the name and title of the land must be marked, with the parish where it lies, and the year it was surveyed.

The stations are marked thus $\odot$ with the numbers 1, 2, 3, 4, &c. shewing their order affixed to them; and

and when it is necessary to return to any of the former stations, in order to close some particular part, that station is set down again with a number prefixed to it, shewing how often the instrument has been placed there; opposite to the number of degrees and minutes of the angles is marked a hook thus $\succ$, or $\prec$ thus, to shew which way the angle turns, either to the left or to the right; for if this should be neglected, it would not easily be known which way the next station lies when the survey is protracted.

P R O B L E M.

405. *To survey with the new improved theodolite.*

Having made the necessary preparations, mentioned before, place the instrument at the first station A, and level it by means of the cross levels and screws underneath; fix the index to 360 degrees on the limb, turn the whole instrument round till the north end of the needle hangs over the flower-de-luce, or 360 degrees, in the box, there fix the limb of the instrument by means of a screw underneath, then discharge the index and turn it about till the vertical hair in the telescope cuts the second station B, there fix the index to the limb by means of a screw, and the north end of the needle will shew the bearing of the line drawn through the first and second stations, which will likewise be cut by the index on the limb: This angle is to be entered into the field-book, under the first station, with the hook $\succ$ at the side of it, as has been mentioned before.

The distance AB between the first and second stations being measured with a chain, in the same manner as has been mentioned in the last problem, as likewise all the offsets from and perpendicular to the line of direction to the several bendings of the boundary, and entered in the field-book, with notes in the margin of the hedges, ditches, stiles, gates, lanes, and houses, which occur between these stations, and are to be expressed in the plan; the instrument is removed to the second station B, and after having levelled it, and so placed as the telescope cuts the first station A; then the limb is fixed in
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that position; the index discharged, and turned about till the vertical hair of the telescope cuts the third station C; there fix the index again and enter the number of degrees and minutes it cuts on the limb; the same angle will likewise be nearly pointed at by one end or other of the needle in the box, if no mistakes have been made.

Care must be taken that the index has not been moved by carrying the instrument from one station to another; this may be known by seeing whether the angle it cuts on the limb is the same with that marked in the field-book: It must likewise be observed, that when the index is moved, the limb of the instrument remains in its place; and when the limb is moved that the index moves with it, so as to point always at the same number of degrees.

The distance between the second and third stations being measured, as well as the offsets, and entered into the field-book, together with all the remarks concerning the objects which are to be inserted in the plan; the instrument is removed to the third station C, where the same operations are performed as at the second; that is, the instrument being laid level, and turned about till the vertical hair of the telescope cuts the second station B; the limb is fixed there and the index discharged, which being turned about till the vertical hair cuts the fourth station D; there it is fixed, and the angle it cuts on the limb is entered into the field-book, as well as the distance and offsets between the third and fourth stations, together with all the necessary remarks.

This being continued all round to the last station, where the instrument is turned to the first in order to close the work, and to find whether there has any mistakes been made in the taking of the angles; the instrument is placed again at the first station, and the angle it makes with the last and second is taken; which, if the same or nearly so as that taken there before, you are certain that no errors have been committed in the taking of the angles.

When there is any occasion to return to a former station,

tion, in order to proceed from thence for closing some particular fields, it is to be entered again in the field-book, as has been mentioned before ; for which reason marks are left in those places where there is occasion to return again.

There is a great nicety required in keeping rightly the field-book, so as to mark distinctly all the particulars which are to be expressed in the plan, and in as short a manner as possible ; and for the better understanding, there ought to be sketches added to the remarks, which may be placed on some other leaf, and referred with notes to the stations and offsets where they have been made.

Thus for example, the house and gardens marked P in the plan, being sketched on a piece of paper, and the dimensions of the building marked on it ; then there is no more required than to take angles from three or four stations to the corners of the house ; as suppose from the stations G, H, I, K, by which the place in the plan of this house will be determined.

In order to survey the river, the instrument is placed at the fourth station D, and the index turned till it cuts on the limb the same angle as at the third station C ; then after the instrument is levelled, and turned about till the vertical hair in the telescope cuts the third station C, there the limb is fixed, the index discharged, and turned about till that hair cuts the station L ; the angle marked by the index on the limb being entered in the field-book under the station D, with the number 2 prefixed to it, to shew that the instrument has been placed twice there ; the distance from D to L being measured, and entered into the field-book, together with the proper offsets and notes, the operations are continued as before.

To render the preceding discourse as intelligible as possible, we shall subjoin a specimen of a field-book, in order to shew how the several particulars are to be entered, and afterwards how the plan is to be laid down from this book, by which it is hoped, that the intelligent reader may survey by himself whenever occasion requires it.

F I E L D.

FIELD-BOOK of part of the manor, &c. in
the parish M, beginning at the lane N, Feb. 16,
1747.

Remarks left.	Offsets	⊙ s.	Offsets	Rem. right
		⊙ 1. A		
hedge	>	75 : 15	SW	bearing
hedgeturns off	100	650		
gate	75	1300		
		1650		dist. from 1st to 2d station
		⊙ 2. B		
hedge	Λ	219 : 30		
hedge	150	770		
	250	1500		dist. from 2d to 3d station
		⊙ 3. C		
hedge	V	245 : 50		
hedge	160	540		
	40	1150		
		⊙ 4. D		
gate	Λ	318 : 30		
to the river	350	150		
gate	30	1550		
		2652		
		⊙ 5. E		
hedge	Λ	310 : 50		
	25	1240		
		⊙ 6. F		
hedge	V	322 : 40		
hedge	30	900		
	25	1920		

Remarks left.	Offsets	⊙ s.	Offsets	Rem. right
		⊙ 7. G		
bearing	△	305 : 0		
hedge	230	750		
hedge	140	1100		
hedgeturnsoff	260	1470		
		⊙ 8. H		
	△	277 : 30		
hedge	140	400		
hedge	30	1430		
		⊙ 9. I		
	△	223 : 10		
hedge	150	1100		
gate, road to the farm	45	1860		
		⊙ 10. K		
		2 ⊙ 4. D		
	△	317 : 0		
		1420		
		⊙ 11. L		
river	30	200		
	150	800		
	460	1400		

N. B. The distances are marked in chains and links; the chain is 66 feet, or 4 poles, divided into a hundred links, invented by one *Gunter*, for which it is called *Gunter's Chain*: The offsets are measured by a rod of ten links; this way of measuring the distances and offsets, is much better than with any other chain of yards or feet; because the several parts are set down with more ease, by means of a scale divided into 100 equal parts.

If there occurs several objects opposite to the same distance, which are to be taken, it will be better to set down that distance several times, by which all confusion is avoided.

406. *To protract the observations on paper, or, which is the same, to draw the plan of a survey from the field-book.*

Draw a line at pleasure, representing a meridian, mark one end of it with a flower-de-luce, representing the north; place the diameter of the protractor on this line, so as the number 360 on the limb be towards the flower-de-luce, and there fasten it with some pins, or short nails: This being done, turn the brass ruler about till the index marks on the limb the angle 75 degrees and 15 minutes, taken at the first station; draw a pencil line along the edge of the ruler, and mark it with the number one; then turn the ruler about till the index points at the angle 219 degrees 30 minutes, taken at the second station; draw a pencil line, which mark with the number 2, turn the index till it points at the angle 245:50, taken at the third station, draw a line and mark it with the number 3.

Continue thus, in laying down all the angles, without moving the protractor; but if there are so many that some of the lines come too near one another, one or more meridians may be drawn parallel to the first, and the protractor placed on them, by this means all the angles may be marked without confusion.

Having marked all the angles, chuse a convenient place for the first station A; then laying the edge of a long parallel ruler on the line marked 1, open it so till the same edge passes through the point A, or the first station, where you draw a line, on which set off the distance 1650 between the first and second stations; that is, 16 chains and 50 links, which gives the second station B; lay the edge of the ruler on the line marked with the number 2, and slide it so as the same edge passes through the point B, where you draw another line, on which set off the distance 1500, that is 15 chains, between the second and third stations; this gives the third station C; lay the edge of the ruler on the line marked 3, and slide it so till it passes through the

the point C, there draw a line, and set off the distance 1150 between the third and fourth stations, which gives the fourth station D: Continue thus to lay down all the stations to the last; then if the last line passes through the first station, and the distance on the paper answers that in the field book, you are certain that no mistake has been made; but if it does not, and all the angles have been found right, on the ground, as we have mentioned in the article of surveying, some error has been committed in measuring the distances.

In order to discover where the mistake lies, protract backwards, till you come to a line which falls on the same line which was found before by the first protracting; then you are certain that you are right; and this will always happen, if the angles are right.

The stations being laid down exactly, you begin at the first to mark the several offsets through which the boundaries are drawn, so as to represent hedges or ditches, according to the notes in the field-book.

Every thing being marked on the plan, it will be proper to see whether nothing has been omitted; and if there is any thing wanted, it must be inserted by guess only, if not very material, or otherwise it must be measured.

P R O B L E M.

407. *To find the number of acres contained in a survey.*

As the boundaries are generally irregular, it is necessary to draw pencil lines, so as to reduce the plan into a rectilineal figure in such a manner, that the parts cut off are nearly equal to those taken in; the reader may easily perceive that this cannot be done otherwise than by guess, but as a geometrical exactness is not required in this work, it may be done so near as to cause no considerable error.

This being done, divide the figure, by pencil lines, into squares, rectangles, and triangles, in the manner most convenient, and find the areas of the several parts separately, add them into one sum, and you will have the contents of the whole.

The

The area of a square or rectangle will be found by multiplying their sides together, expressed in numbers, and that of a triangle is half the (*art. 35*) rectangle of the same base and altitude, or equal to half the product of its base and altitude.

A statute acre is 40 poles long and 4 broad; that is, 4×40 , or 160 square poles; and because the *Gunter's* chain is 4 poles long, an acre is 10 chains in length and 1 in breadth; but this chain is divided into 100 links; 10 chains make 1000 links, which being multiplied by one chain, or 100 links, gives 100000 for the content of an acre expressed by square links. Which shews, *that if the area of a survey is expressed in square links, by striking off the five last figures to the right as decimals, the remaining ones will express the number of acres contained in the survey.*

For example, if 1765436 expresses the number of square links contained in a survey, by striking off the five last figures 65436, we shall have 17 acres, and 65436 decimal parts of an acre, which being multiplied by 4, gives 2.51744 or 2 poles, four of which make an acre, or 2.5 nearly: Therefore the content of this survey is 17 acres and 2.5 poles.

We have not given any figure to shew in what manner an irregular spot of ground may be reduced into a rectilineal figure, because it is quite arbitrary, and the reader who has read the former part of this work will be able, upon occasion, to perform it without any trouble.

P R O B L E M.

408. *To survey or take the plan of a fortification.*

The manner of surveying a fortification does not differ from that described before; it is rather more easy, by reason of its regularity: Of all the different ways of proceeding, the following one appears to me the most simple and practicable. Place the instrument so as that you may take the angle made by the curtain and the flanks, take also the angle from that station to the next, in the same bastion near the other curtain;

measure

measure the curtain and flanks; and if you go round the rampart and continue the same operations near each curtain, you will get the plan of the body of the place: For because the faces of the bastions when produced generally meet the curtains at their extremities; therefore a line drawn through the extremity of the curtain and that of the opposite flank will give the direction of the face, and the line drawn through the extremity of the other flank of the same bastion, and that of the adjacent curtain, will determine both faces.

If the faces produced do not meet the curtains at their extremities, the distance from the extremity of the curtain to the point where the face produced meets it, must be measured; which being done, the rest of the operations will be the same as before.

Having determined the body of the place, the next thing to be done is to find the counterscarp of the great ditch; in order to which, observe whether it meets the shoulder of the bastion when produced, as it generally does, then take the angle it makes with the flank, or the face of the bastion; and if the place is regular, this will be sufficient to determine the breadth of the ditch all round: For, if lines are drawn from the shoulders, so as to make the given angles with the flanks or faces, and from the salient angles of the bastions, perpendiculars are drawn to these lines, they will be the radii of the round part of the counterscarp.

If the place is irregular these angles must be observed all round; and if the counterscarp does not meet the bastions at the shoulders, the distance from that point to the shoulder must be measured, and then the rest of the operations are the same as before.

The ravelins, the counterscarp of their ditches, and the covert way, are all determined in the same manner as the counterscarp of the great ditch; that is, the point where they meet the faces of the bastions, or any other known part, being observed, and the angles they make with these parts measured, then there is no more required than to draw lines thro' the given points, so as to make

the given angles, in order to have the outlines of these works. As to the parapets and ramparts, they are always of the same breadth as those of the body of the place.

If there are any outworks which do not meet the body of the place, when produced, they must be taken from without; the directions of the branches of horn and crown-works must be taken from within the place, and the other parts from without.

The gates of the town, magazines, store-houses, and barracks ought to be inserted in the plan of a fortification; and if they are of any note, their front elevations should be given on a particular scale: In short, every thing belonging to the government should be taken notice of.

P R O B L E M.

409. *To take the map of a country.*

As the survey of a large extent requires much more exactness and care than a small one, it is necessary to have such instruments as will measure the angles to a great nicety; for which reason a sector or quadrant must be had, whose radius is about 5 or 10 feet, and so contrived as to measure seconds: For a few seconds, more or less, in large surveys, will cause considerable errors at the end of the work.

Having proper instruments for such an undertaking, a large plain is pitched upon, nearly in the middle of the country, so as a base of at least a mile long may be measured; this should be done with the greatest exactness possible, and with which a triangle is to be formed with some steeple in a town; then the distance from one end of this base to the town is to be found by trigonometry; and with it, and any other adjacent towns, another triangle is to be formed so as to get the distance between these two places: And it is this last distance which is to be taken for the base, to form triangles from all the different towns. As the exactness of the whole survey depends on this base, there cannot be too much caution used to measure it; for which reason I would chuse to form some other triangle by means of a different

rent base; then if the same measure is found for this distance, one may depend on its exactness.

Having the exact distance between any two places, it will serve as a common base to form from thence triangles to all the circumjacent places; and the distances between these places, as bases to form other triangles to some others; and in this manner a chain of triangles may be formed, so as to go thro' all the considerable places of the whole country.

It may sometimes happen, that from two stations, another cannot be seen; in which case a pyramid of wood, or high poles are erected on some hill, from whence the triangles may be carried on; when the distances are very great, fires are lighted upon eminences, to guide the observations by, especially if the objects cannot otherwise be seen.

As there cannot be too many cautions used, in order to avoid errors, it is necessary to find the latitudes of all the principal towns, by observations. This may be done by observing the altitude of the sun at 12 o'clock exactly; and by a table of the sun's declinations, you find the elevation of the equator in that place; and the complement of this angle will be the latitude required. There are various methods for finding the latitude of a place, which those who are employed in such a work as this, ought not to be ignorant of.

Having determined the exact positions of all the principal places, the next thing to be done is to determine the high-roads, and the courses of great rivers: This may be done with a theodolite, in the same manner as has been shewn in prob. II. only there is no occasion to take the offsets so frequently; it will be sufficient to take the principal bendings. The plans of all the remarkable places should be taken, and it should particularly be observed, on which side of the principal streets the church stands, because the distances are determined by their positions.

If there are any considerable woods, their limits should be determined; as likewise all the great roads which

pass through them; high hills must likewise be marked in the map; and if they are of an extraordinary height or size, sketches are to be made of them, to be placed by themselves in some corner of the map.

As in a country the variety of objects is infinite, so it is impossible to give particular directions concerning the manner of observing them: It requires great skill in making the observations, and to chuse proper places for the stations: But as works of this nature are never undertaken by unexperienced people, we hope what has been said on this subject will be sufficient to suggest such means to them, as will be a great help in carrying on the work with success.

SECT. III. Of LEVELLING.

THERE are two sorts, the water and spirit, or air-level; the first is commonly made of a glass tube of about two feet long, sometimes shorter, of an inch diameter, the ends of which are bent at right angles for the length of two inches or thereabouts, and stopt with brass stoppers, to prevent the water from running out.

This level is supported by a three legged staff, in the same manner as the theodolite, fixed to the middle of the tube with a brass ring.

In the country, where a glass tube is not easily to be had, they make a tube of tin about the same dimensions, to the ends of which are fixed two viols with sealing wax, the bottoms being broke, so as the water in the tube may have a free communication with that in the viols.

P R O B L E M.

410. *To level with the water level.*

Place your level at a convenient distance from the place of your setting out, so as the water rises in both ends of the viols; direct your eye over the two surfaces of the water, towards the first object, where somebody must stand with a white sheet of paper, to raise or depress it, according to the motion of your hand, along the station-

station-staff, till the black line drawn across the paper is cut by the line of observation; this being done, the height of this line from the ground is measured, and you go to the other end of the level and look forward, to the next station, where the height of the black line from the ground is also measured.

The instrument being carried forward, as much beyond the second station as you can conveniently see, and another station placed farther, you make the same observations backwards to the last station, and forward to the next; and you continue in the same manner till you come to the object, whose level, in respect to the first, is wanted.

This done, all the heights taken backward being marked in one column, and all those taken forward in another, then the sum of the one subtracted from the other, will give the difference between the levels of the first and last object: If the sum of the heights forward exceeds the sum of those backward, the first object is higher than the second, by the difference between these sums, and on the contrary.

Because of the roundness of the earth, the level found between any two objects in the preceding manner is not the true one, for which reason it is called the apparent level.

P R O B L E M.

411. *To find the difference between the true and apparent levels of any two places.* Plate XIII. Fig. 3.

Let O be the center of the earth, OA, OD, any two radii, AB a tangent to the surface ADE, at A: Now if the level be placed at A, it will give the apparent level AB, between the places A and D; whereas the true level is the arc AD: So that the apparent level exceeds the true one by the line BD: For which reason this line must be found and subtracted from the apparent level, in order to get the true level between the places A and D.

By the property of the circle, we have (*art.* 116) $2OD + DB \times DB = AB^2$; or, because the diameter of the earth is so great in respect to the line BD, at any

distance not exceeding 30000 yards, that $2OD$ may safely be taken for $2OD + DB$, without any sensible error, as may be found by calculation: Therefore we have $2OD \times DB = AB^2$, according to this supposition; which shews, *that the difference between the true and apparent level is equal to the square of the distance between the objects, divided by the diameter of the earth.*

The diameter of the earth is 653869 toises or fathoms, *French* measure, according to Mr. *Picard*, or 13077388 yards; and as the *English* foot is to the *French* royal foot as 107 to 114, according to the most accurate comparison, the diameter of the earth will be 13932824 yards *English* measure.

Now suppose we want to know the difference between the true and apparent level, at a distance of an *English* mile, or 1760 yards; the square of this number is 3097600, which being divided by the diameter of the earth 13932824 expressed in the same measure, gives 0.222 decimals of a yard; or multiplying this number by 36, the number of inches in a yard, gives 7.999, or 8 inches for the said difference. This agrees with what other authors say on this subject.

In order to save the trouble of computing every time the difference between the true and apparent level, we shall insert the following table of corrections from a distance of 210 yards to 1000, expressed in inches and decimal parts of an inch.

A T A B L E, shewing the corrections to be made between the true and apparent level.

210	220	230	240	250	260	270	280	290	300	Dist.
.11	.12	.14	.15	.16	.17	.19	.20	.22	.23	Cor.
310	320	330	340	350	360	370	380	390	400	Dist.
.24	.26	.28	.30	.32	.33	.35	.37	.39	.41	Cor.
410	420	430	440	450	460	470	480	490	500	Dist.
.43	.45	.47	.49	.52	.54	.57	.59	.61	.64	Cor.
510	520	530	540	550	560	570	580	590	600	Dist.
.67	.70	.72	.75	.78	.81	.84	.87	.90	.92	Cor.
610	620	630	640	650	660	670	680	690	700	Dist.
.96	.99	1.02	1.06	1.09	1.12	1.15	1.19	1.23	1.27	Cor.

710	720	730	740	750	760	770	780	790	800	Dist.
1.30	1.34	1.38	1.41	1.45	1.49	1.53	1.57	1.61	1.65	Cor.
810	820	830	840	850	860	870	880	890	900	Dist.
1.69	1.74	1.78	1.82	1.86	1.91	1.95	2.00	2.04	2.09	Cor.
910	920	930	940	950	960	970	980	990	1000	Dist.
2.14	2.19	2.23	2.28	2.33	2.38	2.43	2.48	2.53	2.58	Cor.

By means of this table the apparent level may be corrected, for any distance (not exceeding 1000 yards: If the distance is not to be found exactly in this table, that which comes nearest to it must be taken, for the difference will hardly be sensible.

Because most surveyors use the *Gunter's* chain in surveying and levelling, and compute the corrections by the distances expressed in chains, we shall insert the following table where the distances are expressed in chains, and the corrections in inches and decimal parts of an inch.

5	6	7	8	9	10	11	12	13	14	Dist.
.03	.04	.06	.08	.10	.12	.15	.18	.21	.24	Cor.
15	16	17	18	19	20	21	22	23	24	Dist.
.28	.32	.36	.40	.45	.50	.55	.60	.67	.72	Cor.
25	26	27	28	29	30	31	32	33	34	Dist.
.78	.84	.91	.98	1.05	1.12	1.19	1.27	1.35	1.44	Cor.
35	36	37	38	39	40	41	42	43	44	Dist.
1.53	1.62	1.71	1.80	1.90	2.00	2.11	2.21	2.30	2.42	Cor.

These tables are computed in an easy manner by means of logarithms; for the logarithm of the diameter of the earth 13932824, expressed in yards, is 7.14404; no more figures are necessary; which being constantly subtracted from double the logarithms of the distances, the differences will be the logarithms of the corrections expressed in decimal parts of a yard; which being multiplied by 36, the number of inches in a yard, gives the corrections in inches and decimals of an inch.

Having explained the nature and use of the water level, as likewise in what manner the apparent level is corrected, we shall now give the description of the spirit level, with the manner of correcting and using it.

Of the SPIRIT LEVEL.

This level is a glass tube of about 6 inches long; half an inch diameter, filled with spirits almost quite full, so as to leave a little air in it; this tube is fixed to a telescope of about two feet long, with cross hairs in it; and the whole instrument is supported by a three-legged staff, in the same manner as the plain table or theodolite.

The station staves used in levelling are two rulers, each about 5 feet long, sliding along each other by means of two brass ferules fixed to their ends; these rulers are divided into inches and tenths of inches; at one end is fixed a board of 6 inches long and 4 broad, called Vane, which slides likewise along the ruler, and serves to paste a piece of paper upon, one half white and the other black, placed horizontally.

P R O B L E M.

412. *To rectify the spirit level.*

The first thing to be done is to fix the intersection of the cross hairs exactly in the axis of the telescope; in order to which observe a point as far off as may be seen distinctly; then turn the telescope round its axis, till the lower part comes uppermost; and looking again to see whether the intersection of the cross hairs cut still the same point; if it does, the hairs are rightly fixed; but if not, slide the horizontal hair so as their intersection cuts in a point between the two former, and turn the telescope round its axis till the upper part is again the lower.

Observe again the point where the cross hairs cut an object; if it is the same as the last observed, the telescope is rectified; if not, slide the horizontal hair so as its intersection cuts a point between the two last observed, and continue in this manner till the intersection of the cross hairs cuts always in the same point which ever way it is turned.

The next thing to be done is to bring the level parallel to the axis of the telescope; for which, set the instrument

strument so as that the bubble in the glass tube be in the middle; observe through the telescope where the cross hairs cut an object; then take it off, and turn the instrument round its axis, so that the part which was next to the eye be towards the object; place the telescope on the instrument, and see whether the intersection of the cross hairs cut the same point as before; if not, raise one end of the telescope, or the level, by means of a screw made for that purpose; then set the instrument again, so as the bubble be in the middle of the glass tube, and observe where the cross hairs cut an object; take off the telescope; turn the instrument round to its former place, and see if the cross hairs cut still the same point; if not, raise or depress the telescope or level, and continue the same operations till the instrument is rectified.

When the level is fixed to the cross bar which supports the telescope, it is much better than when it is fixed to the telescope itself; for the brass tube being very thin, the screws cannot have hold enough to keep it firm: the forks which support the telescope, seem to me ill contrived, because the screws that fix them are not sufficient to support such a weight as the telescope and the forks together; besides, I think that the telescope does not lie so steady as it ought to do.

P R O B L E M.

431. *To find the difference between the levels of any two places, P and S, which are at a great distance from each other. Plate XIII. Fig. 2.*

Having found a convenient place between the two first stations P and Q to place the instrument, raise it so as the air-bubble be exactly in the middle of the glass tube; direct the telescope to the first station P, where a Person stands to raise or depress the vane, according to the motion of your hand, till the horizontal line which separates the black part from the white is cut exactly by the intersection of the cross hairs, which suppose to be at number 1; then the height of that point from the ground is marked in a pocket-book, and you direct the

in.

instrument to the second station Q; seeing that the bubble remains in its proper place, if not, the level must be rectified; then observe the place 2, where the cross hairs cut that station; and having taken the height of that point from the ground, and marked it in the book, the distances from the level to each station are likewise measured; this done, the instrument is removed to some convenient place B, between the second Q and third station R; set the level so as the air-bubble be in the middle of the glass tube, and direct the telescope back to the second station Q, the point 3, where the intersection of the cross hairs cut the station, being observed, and its distance from the ground entered in the book, turn the instrument and direct it to the third station R, seeing that the bubble remains in the middle of the tube, the point 4, where the cross hairs cut that station being observed, and its distance from the ground entered; the distances from the instrument to the stations being also measured and entered, the instrument is removed to a convenient place C, between the third and fourth stations, where the same operations are performed as before, and are continued to the last station. This done, set all the heights observed backwards in one table, and all those observed forwards in another; do the same thing in regard to the distances, from the instrument to the stations; correct the apparent levels by the foregoing tables, then subtract the least sum of all the heights corrected of one side, from the greater sum of all the heights corrected of the other, and the remainder will be the difference between the heights required of the two places. If the sum of all the heights forward exceeds the sum of all those backward, the first place is the highest, as has been observed before.

N. B. Mr. *Burton*, mathematical instrument-maker, facing the New Church in the *Strand*, makes the most compleat levels that have been made.

E X A M P L E.

Dist. *Heights* *Cor.*

A2 250	P1 10	9.83
B3 200	Q3 3	2.90
C5 360	R5 8	7.69

Dist. *Heights.* *Cor.*

A2 350	Q2 6	5.69
B4 460	R4 16	15.51
C6 300	S6 13	12.77

First sum 20.43

Second sum 33.97

Difference 13.54

Whence the first place P is 1920 yards distant from the second S, and is higher by 13.54 inches.

It may be observed, that if the distances from the level to both stations are equal, there requires no correction; but it is not always convenient to place the instrument in this manner.

It is not necessary that this level should be in a right line with the stations, as in the water level; for whether it makes any angle therewith or not, it signifies nothing.

Levelling is of great use in draining marshes, making rivers navigable, and bringing waters to a town from some river or spring: in this case it has been found, that if the water has a descent of about 5 inches in a mile, it is sufficient; and when it is much more, the
current

current becomes so rapid as to destroy the banks of the river.

SECT. IV. Of MENSURATION.

BY Mensuration is commonly meant the manner of measuring and computing the dimensions of the several parts of a fortification, or of any other building both in civil and military architecture: every country has its particular measures; here in *England* the foot, yard, fathom, or pole, are the measures generally used, which being well known, need no farther explanation.

Mensuration is either lineal, plane, or solid; the lineal is used in surveying and levelling; the plane, in finding the contents of fields, floors, paintings, glasses, and, in general, of every thing where the superficies are only required; and the solid, in measuring walls, timbers, excavations, &c.

It has been proved (*art. 40*) in geometry, that parallelograms or rectangles which have equal bases and equal altitudes, are equal: hence arises the common rule, that the area of a rectangle or square is found by multiplying their sides together. It has also been (*art. 35*) shewn, that a triangle is equal to half the rectangle of the same base and altitude: and as all rectilineal figures may be reduced into triangles, it is evident, that their areas or contents may be found by the same rule.

Though the reason of this rule is obvious, yet why one length multiplied by another should produce square superficies, has either been much neglected, or unknown to most authors and teachers of arithmetick, who generally propose the erroneous question of multiplying money by money.

Plate XIII. Fig. 4. In order to make this plain, suppose the sides of a square ABCD to be divided into any number of equal parts, such as AE, by lines parallel to the sides; then, whatever the number of parts are into which AD is divided, it is evident, that the rectangle ABFE, whose base AE is one of them, will contain just as many small squares; that is, if AD be

di-

divided into six equal parts, the rectangle AF will contain six small squares: it is no less evident, that the square ABCD contains also as many rectangles, such as AF, as the sides are divided into equal parts: therefore, if the number of squares contained in the rectangle AF, be multiplied by the number of the rectangles contained in the great square ABCD, the product will express the number of small squares contained in the great one AC.

If ABCD be a rectangle, and the side AD is to the side AB, as 10 to 12; it is evident, that if the base AE be the tenth part of AD, the rectangle AF will contain 12 small squares, whose sides are equal to AE; and the figure ABCD will contain ten rectangles such as AF. Consequently, if the number 12 of the squares contained in AF be multiplied by the number 10 of the rectangles, the product 120 will express the number of small squares contained in the rectangle ABCD.

If the base of a triangle be 8, and the perpendicular 12, then the rectangle of the same base and altitude will contain 8 times 12, or 96 small squares, whose sides are unity; and as the triangle is half this rectangle, half of 96, that is 48, will express the number of the squares contained in the triangle.

Hence, a square foot contains 12 times 12, or, 144 square inches; a square yard 3 times 3, or 9 square feet; and a square fathom 6 times 6, or 36 square feet: therefore, if the area of any figure be expressed in square inches, by dividing it by 144, the number of square inches in a foot, the quotient will express that area in square feet; if the area be expressed in square feet, by dividing it by 9 or 36, the quotient will express the number of square yards or fathoms contained in that area.

This method of finding the areas of figures is easy, when the dimensions are expressed by numbers of the same denomination; but when they are expressed by feet, inches, and decimal parts of an inch, they must be reduced into the lowest denomination; and afterwards,

wards, the product divided by the number of square inches in a foot, yard, or fathom, according as the area is to be expressed in feet, yards, or fathoms; by which the operations become very tedious and troublesome.

It is on this account that the following method is used, which is both shorter and easier; and is, that the parts of a lower denomination are adapted to that of a higher, in the same manner as in lineal mensuration: That is, 12 inches make always a foot, 3 feet a yard, or 6 feet a fathom. Which shews, that all the parts of the same superficies are reduced to the same height; so that a superficial foot, referred to yards, is a rectangle whose base is a foot and height a yard; an inch referred to yards, a rectangle whose base is an inch and height a yard; if an inch or foot be referred to fathoms, their heights are a fathom; and so on. Consequently, a foot or inch multiplied by a yard, gives a superficial foot or inch of a yard; or if a foot or inch be multiplied by a fathom, the product will be a superficial foot or inch of a fathom.

Fig. 5. The same thing is to be understood in regard to solid contents; for if the cube AB be divided into any number of equal parts, such as ADC by planes, parallel to one of the sides AC, and ADC be the sixth part of a cubic fathom, it is called a solid foot of a fathom; therefore a solid foot of a fathom has a square fathom for its base, and a foot in height; in the same manner a solid foot or inch of a cubic yard has a square yard for its base, and a foot or inch in height. Consequently, a square yard multiplied by a foot or an inch, gives a solid foot or inch of a yard; and a square fathom, multiplied by a foot or an inch, gives a solid foot or inch of a fathom.

EXAMPLE I.

414. *A floor being 6 yards 2 feet 6 inches long, and 5 yards 2 feet 4 inches broad, what is the content of the floor expressed in yards?*

Say

Say 5 times 6 inches gives 30 inches, or 2 feet 6 inches, set down 6 inches and carry two; then 5 times 2 feet gives 10 feet and 2 carried is 12, or four yards; set down 0 and carry 4; and 5 times 6 gives 30, and 4 carried, is 34 yards.

$$\begin{array}{r}
 6 \ 2 \ 6 \\
 5 \ 2 \ 4 \\
 \hline
 34 \ 0 \ 6 \\
 4 \ 1 \ 8 \\
 2 \ 3\frac{1}{2}
 \end{array}$$

Now because 2 feet are 2 thirds of a yard, begin with the yards and say, the two thirds of 6 yards are 4, set down 4 yards; the two thirds of 2 feet are 1 foot 4 inches, set down 1 foot and carry 4; and the two thirds of 6 inches are 4, and 4 carried make 8, set down 8 inches.

As 4 inches are the sixth part of 2 feet, or 24 inches, we are to take the sixth part of the last product for 4 inches, by saying, the sixth part of 4 yards or 12 feet is 2 feet, the sixth part of a foot, or 12 inches, is 2 inches; and the sixth part of 8 inches is $1\frac{1}{3}$, which, with the 2 carried, makes $3\frac{1}{3}$ inches.

Now all these products being added, by carrying 1 foot for 12 inches, and 1 yard for 3 feet, we shall have 39 yards 1 foot and $5\frac{1}{3}$ inches for the content of the floor.

DEMONSTRATION.

We have shewn that yards multiplied by feet and inches give superficial feet and inches of a yard; and because 1 yard multiplied by 6 yards 2 feet 6 inches, gives the same number of yards, feet, and inches, it is evident, that 2 feet, which are two thirds of a yard, will give also the two thirds of these numbers; and by the same reason 4 inches, or one sixth part of 2 feet, will give the sixth part of the product of 2 feet.

EXAMPLE II.

415. *If a piece of painting is 12 yards 1 foot 9 inches long, and 7 yards 1 foot 4 inches broad, how many yards does this superficies contain?*

Say 7 times 9 inches is 63 inches, or 5 feet 3 inches, set down 3 inches and carry 5; 7 times

1 foot

1 foot is 7, and 5 carried is 12, set down 0, and carry 4 yards; 7 times 2 is 14, and 4 carried is 18, set down 8 and carry 1; 7 times 1 is 7, and 1 carried is 8.

12	1	9
7	1	4

88	0	3
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As 1 foot is a third part of a yard, say one third of 12 yards is 4, set down 4 yards; one third of one foot, or 12 inches, is 4, set down 0 and carry 4; one third of 9 is 3, and 4 carried is 7, set down 7 inches.

4	0	7
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1	1	2½
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93	2	0½
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And because 4 inches are one third of 1 foot, or 12 inches, take one third of the last product, by saying one third of 4 yards is 1 yard and 1 foot, set down 1 yard, and carry 1 foot to the feet; and one third of seven inches is 2 ⅓ inches, set down this to the inches; add all these products together, and there will be 93 yards 2 feet and ⅓ of an inch.

N. B. It must be observed that floors, wainscoting, painting, and window glasses, are all measured by square yards; and their dimensions are taken by applying a thread close to all the moldings; and the sides of a room are taken as if there were no vacancies; and the openings of the windows are taken separately and subtracted from the first sum, in order to have the true content of the work.

EXAMPLE III.

416. *Let a wall be 50 fathoms 3 feet long, and 3 fathoms 4 feet 6 inches high; how many square fathoms does this wall contain?*

Say 3 times 3 feet is 9 feet, or 1 fathom and 3 feet, set down 3 feet, and carry 1 to the fathoms, &c. then as 4 feet are the two thirds of a fathom, and two thirds are not easily taken in large numbers, I take one third for 2 feet, and set the product down twice; and since 6 inches are one fourth of 2 feet, or 24 inches, take one fourth of the product of 2 feet, and the sum of all these products,

50	3	0
3	4	6

151	3	0
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16	5	0
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16	5	0
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4	1	3
---	---	---

189	2	3
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gives

Sect. 4. MATHEMATICS.

gives 189 fathoms 2 feet and 3 inches for the content required.

E X A M P L E IV.

417. If an area is 24 fathoms 2 feet 9 inches one way, and 7 fathoms 9 inches another; how much is the content expressed in fathoms?

Here I multiply the upper factor by 24 fathoms, in the usual manner; then as there are no feet in the multiplier, I suppose 2 feet, which gives 8 fathoms and 11 inches; this product must be crossed, as not belonging to the area; then I take for 6 inches the fourth part of this product, and for three inches half the product of 6; and the sum, exclusive of the product of 2 feet, gives 174 fathoms, 1 foot, and $7\frac{1}{2}$ inches, for the content of the area required.

24	2	9
7	0	9
171	1	3
8	0	11
2	0	$2\frac{1}{2}$
1	0	$1\frac{1}{2}$
174	1	$7\frac{1}{2}$

E X A M P L E V.

418. Let the dimensions of a solid be such as expressed in the margin, to find its content.

Here you multiply any two dimensions together, as before; suppose the second and third, which gives 86 yards, 0 feet, 1 inch, and this sum is again multiplied by the first dimension 20 yards, 2 feet, 6 inches, in the usual manner, and the product will be as in the margin.

It must be observed, that instead of taking the two thirds for 2 feet, we have taken one third for one foot, and set the product down twice, as being easier in large numbers; we have also neglected the fractions of inches, as being of no manner of consequence.

<i>Yds.</i>	20	2	6
	12	1	4
	6	2	9
	74	2	0
	8	0	10
	2	0	2
	1	0	1
	86	0	1
	20	2	6
	1720	1	8
	28	2	0
	28	2	0
	14	1	0
	1792	0	8
	E X A M.		

419. Let there be an excavation of earth, whose dimensions are expressed in the margin; to find its solid content.

Here the first denomination is fathoms. and the first and second dimensions are multiplied together, then the product by the third.

It must be noted when a multiplier is of more than one figure.

It is more expeditious to multiply them separately, and then reduce them to a higher dimension; as here 25 fathoms multiplied by 7 inches gives 175 inches, which being divided by 12 gives 14 feet and 7 inches; the inches are set down, and the feet carried to the place of feet in the usual manner.

In the application of mensuration to masonry, it is the custom to reduce the thickness of the wall to a brick and a half; for which reason, having found the superficial contents, say as 3 is to the thickness of the wall expressed in half bricks, so is the superficial content of the wall, reduced to a brick and a half thick.

Fath.	10	1	7
	25	2	8
	6	1	4
	256	3	7
	3	2	6
	1	0	10
	261	0	11
	6	1	4
	1566	5	6
	43	3	1
	14	3	0
	1624	5	7

S E C T. V.

METHOD for Measuring SURFACES and SOLIDS, by means of their centers of gravity.

D E F I N I T I O N S.

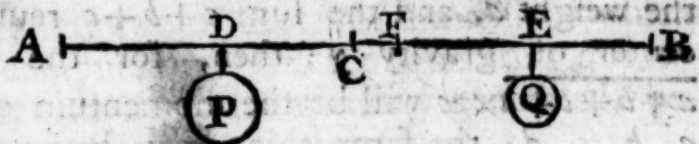
1. **T**HE center of Gravity, is a point whereby a body being supported or suspended, all its parts will remain at rest.

2. *Momentum* of a body is its tendency towards a plane, in respect to its distance from it.

Tho' lines and surfaces have no real weight, yet as their centers of gravity are very useful, their contents will be considered instead of weights.

420. Hence

420. Hence, the center of gravity of a right line AB is the point C which bisects it: For were it supported by that point,



there would be no reason for its inclining more one way than another. If another point F be taken in that line, and the unequal parts AF, FB, bisected in D, E; then as $AC = 2DF - CF$, $CB = 2FE + CF$, and $AC = CB$, by supposition, we have $2DF - CF = 2FE + CF$, or $2DF = 2FE + 2CF$, by equality of ratios, and $DF = EF + CF$, by division; hence $DF = CE$ and $CD = EF$: Therefore $CD : CE :: (2EF : 2DF) FB : AE$.

421. Hence, if two weights P, Q, proportional to the lines AF, FB, were suspended at the centers of gravity D, E, they would have the same effect upon the point C, as these lines re-united into their centers of gravity: Therefore $CD : CE :: Q : P$, by the last: Whence, if two weights suspended by an inflexible line, are reciprocally proportional to their distances from the point fixed, or which is the same, if their momentums are equal, they will remain at rest.

T H E O R E M.

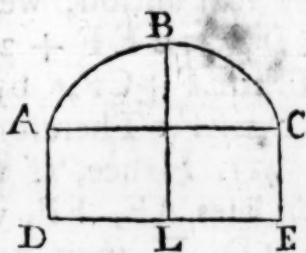
422. The sum of the momentums of any number of weights a, b, c, d , is the same as if they were reunited into their common center of gravity. Fig. 6.

From these weights draw lines perpendicular to the plane AB, as also from the center of gravity x of the weights a, b , and bn , am perpendicular to ox ; then the similar triangles amx, bnx , give $(ax : bx ::) mx : nx :: b : a$, by the last, or $a \times mx = b \times nx$; but $ao = mo$, $bo = no$, and $mx = xo - ao$, $nx = bo - xo$: hence $a \times xo - ao = b \times bo - xo$, or $a \times ao + b \times bo = a + b \times xo$, by transposition.

Join the weight c to the center of gravity x , and let y be the center of gravity of the weight c , and $a + b$, reunited into their center of gravity x ; then by what has been proved before, $a + b + c \times yo$, will be the momentum of

of the weights a, b, c . Join the weight d to the center of gravity y , and let z be the center of gravity of the weight d , and the sum $a+b+c$ reunited into the center of gravity y ; then, for the same reason, $a+b+c+d \times zo$ will be the momentum of the weights a, b, c, d ; the same thing may be proved, whatever number of weights may be: only observe, that if any weights are placed on the contrary side of the plane AB, in respect to the others, their momentums must be made negative.

423. Hence, if the figure ABC, revolves about a line DLE as an axis, and the arc ABC, be supposed divided into an innumerable equal small parts or elements; the sum of all the products of these elements, each multiplied by its distance from the axis DE, will be equal to the product of that arc multiplied by the distance of its center of gravity from that axis: or if instead of these distances or radii, we take the circumferences they describe in the rotation; *the surface described by the arc ABC, will be equal to the rectangle made by that arc and the circumference described by its center of gravity.*



424. Again, if the area ABC, be supposed divided into an innumerable small parts or elements parallel to the axis DE: the sum of the products of all these elements, each multiplied by its distance from the axis DE, will be equal to the product of this area, multiplied by the distance of its center of gravity from that axis; or if instead of these distances or radii, we take the circumferences they describe in the rotation; *the solid described by the area ABC, about the axis DE, will be equal to the solid made of that area and the circumference described by its center of gravity.*

GENERAL RULE.

425. *The content of a surface or solid described by a circular motion about any line as an axis, is always equal to*

the

the product of the generating quantity multiplied by the circular arc described by its center of gravity.

426. Hence, any two of the three dimensions which enter into the computation, viz. the generating quantity, its center of gravity or the quantity generated, being given, the third may be always found.

P R O B L E M.

427. *To find the center of gravity, of a rectangle BD, or that of a right angled triangle AHB, from the side HB.*
Fig. 7.

Suppose a cylinder described by the rectangle HC, and a cone by the triangle AHB, both about the axis HB; let the radius of the base AH or HD, be a , half its circumference c , and the axis HB, b ; then will ac express the area of the base, and abc the content of the cylinder, which divided by the rectangle ab , gives c for the circumference described by the center of gravity, this being half that of the base, its radius is also one half of HD. The content of the cone is $\frac{1}{3}abc$, and its generating plane $\frac{1}{2}ab$, and therefore $\frac{1}{3}abc$, divided by $\frac{1}{2}ab$, gives $\frac{2}{3}c$ for the circumference described by the center of gravity: which being one third of $2c$, gives $\frac{1}{3}AH$, for the distance required from the axis HA. By the same reason the distance of the center of gravity of the rectangle from HD is equal to half of HB, and that of the triangle one third of HB.

E X A M P L E.

428. *To find the content of a right cylinder P, from which a frustum of a cone has been subtracted.*

Let ABCD be the section of this solid, or the generating plane; GL the axis of rotation; draw BH parallel to CD; then the solids described by the triangle ABH, and the rectangle HC, must be found separately, and their sum will be the content required.

If AG be 4, AH, 6, HD, 8, and DC, 12, then the triangle ABH will be half of 6×12 or 36; and the distance from its center of gravity to the axis of rotation GL is equal to AG (4) and to two thirds (*art. 425*) of

O 3

AH

AH (6) or 8; and if $7:22::16:50\frac{2}{3}$, this fourth term will be the circumference described by the center of gravity. Therefore 36, multiplied by $50\frac{2}{3}$, gives $1810\frac{2}{3}$ for the content of this solid.

The rectangle HC is equal to 8×12 , or 96, and the distance from its center of gravity to the axis GL of rotation, equal to GH (10), and to half of HD (8) or 14; and if $7:22::28:88$, this fourth term multiplied by 96 gives 8448 for the content of the solid. Consequently, the sum of $1810\frac{2}{3}$ and 8448 gives $10258\frac{2}{3}$ for the content required.

E X A M P L E.

429. *To find the content of a frustum of a right cone Q, from which a cylinder has been subtracted.*

Let ABCD be a section through the axis of this solid, EF the axis of rotation: If ED be 3, and the rest as before, then the center of gravity of the rectangle from the axis EF is equal to ED (3), and to half of DH (8) or 7; therefore, the circumference described by this center of gravity is 44, and the rectangle 96, multiplied by 44, gives 4224 for the content of the solid described by that rectangle. The distance from the center of gravity of the triangle ABH, to the axis EF of rotation, is equal to EH (11), and to one third of AH (6) or 13; and if $7:22::26:81\frac{1}{3}$, then $81\frac{1}{3}$, multiplied by the triangle 36, gives $2941\frac{1}{3}$ for the content of the solid. Consequently, the sum of $2941\frac{1}{3}$ and 4224 gives $7165\frac{1}{3}$ for the content required.

These two last examples are particularly useful in the measuring of the orillons and retired flanks.

E X A M P L E.

430. *To find the content of the solid described by the rotation of the semi-circle ABC, about the tangent AT, perpendicular to the diameter AC. Fig. 8.*

It is evident that the center of gravity of the semi-circle is somewhere in the radius OB, parallel to AT, since that radius bisects all the lines that can be drawn parallel to the diameter AC. If therefore the radius AO is 14, the circumference of that radius will be 88, and

and the area ABC will be 14×22 , or 308, which being multiplied by 88, gives 27104 for the content required.

Whether the axis AT of rotation touches the circle, or is at some distance from it, the content of the solid described about that axis is always found in the same manner; provided the circumference described by the center of gravity be rightly found, that is, according to its distance from the axis of rotation.

But if the solid described by the semi-circle about the tangent TB, parallel to the diameter AC, be required, the distance of the center of gravity from that line must be known.

E X A M P L E.

431. *To find the center of gravity L of the semi-circle.*

If the radius AO be a , the semi-circumference c ; then the area of the circle (art. 173) will be ac , and the solid content of the sphere (art. 216) $\frac{4}{3}aac$: Now as this solid is likewise equal (art. 425) to the product of the generating quantity ABC, multiplied by the circumference described by the center of gravity L, if the solid $\frac{4}{3}aac$ be divided by the area $\frac{1}{2}ac$ of the semi-circle, we shall have $\frac{8}{3}a$ for the circumference described by the center of gravity; and if $2c : a :: \frac{8a}{3} : \frac{4aa}{3c}$, this fourth term will be the distance required from the center O of the circle.

E X A M P L E.

432. *To find the content of the solid described by the semi-circle about the tangent TB, parallel to the diameter AC.*

The same thing being supposed as before, if OL $\left(\frac{4aa}{3c}\right)$ be subtracted from OB (a) we shall have $a - \frac{4aa}{3c}$

equal to LB; and if $a : 2c :: a - \frac{4aa}{3c} : 2c - \frac{8}{3}a$; this fourth term will be the circumference described by the center of gravity about the axis TB; which being multiplied by the area $\frac{1}{2}ac$ of the semi-circle, gives $\frac{1}{2}c \times 3c - 4a$ for the content of the solid required.

If the radius AO (a) be 7, then the semi-circumference c will be 22; and therefore $7 \times 22 \times 12\frac{1}{2}$ or 1950 $\frac{1}{2}$ will be the content of the solid.

This last example is useful in finding the solid content of a vault, called by the *French*, Bonnet de Pretre.

What has been said here is sufficient to shew the manner of applying the center of gravity to the finding the contents of surfaces and solids, in a most compendious and concise manner; some part of its use will appear in what follows.

MENSURATION *applied to* FORTIFICATION.

Plate XIV. Fig. 10. When the plan of a fortification is traced, there is a line which is called the Principal, or Majistral, on which the length of the parts are set off; and it is this line which is represented by the cordon, marked by the letter C in the profil: For instance, when it is said that the faces of a bastion are 50 toises, it is to be understood near the cordon, where the slope of the wall begins.

Fig. 12. In order to measure the revetement of the face of the bastion AD, we must consider the profil (fig. 10) ABCD, which we shall suppose, according to Mr. *Vauban*, to be 30 feet, or five toises high from the foundation AD to the cordon BC, 5 feet at BC, and 11 at AD; adding these numbers together we get 16 feet for their sum, half of which, 8 feet, or 1 toise and 2 feet, being multiplied by the height AB (5) gives 6 toises 4 feet square for the area ABCD.

The foundation ADIH must be measured by itself, and cannot be determined here, as depending on the nature of the ground, but being always rectangular may be computed easily; the upper part of the wall EFGC must likewise be measured separately.

Now suppose the twelfth figure represents the wall from the foundation to the cordon; from the inner angle A, draw AB, AC, perpendicular to the faces, thro' which suppose to pass two vertical planes; then the angular part cut off will be a frustum of a pyramid, like
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Fig. 1.

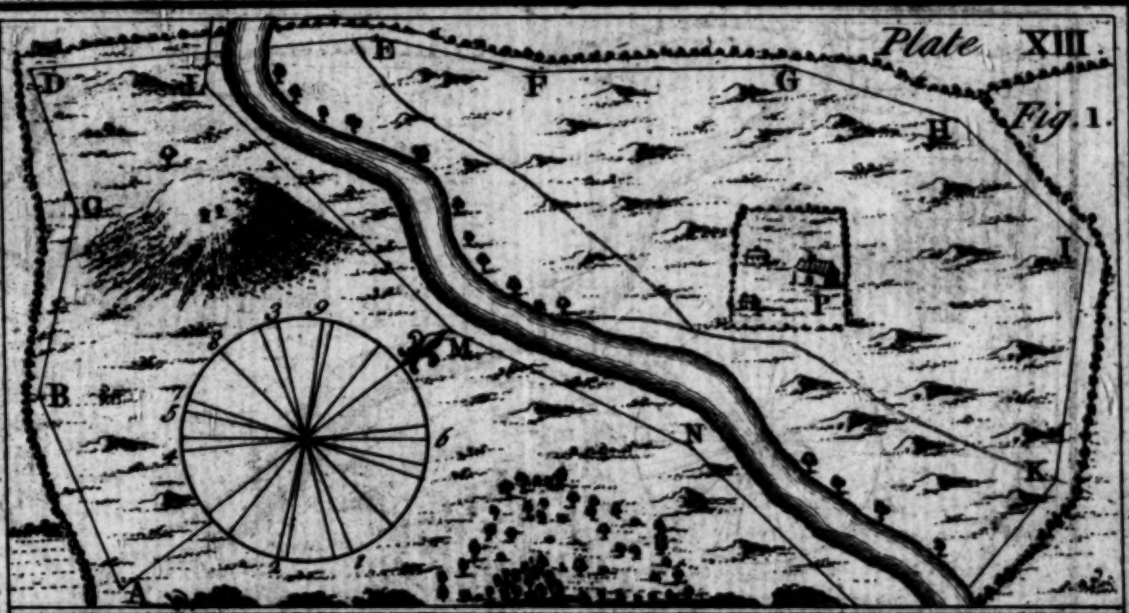
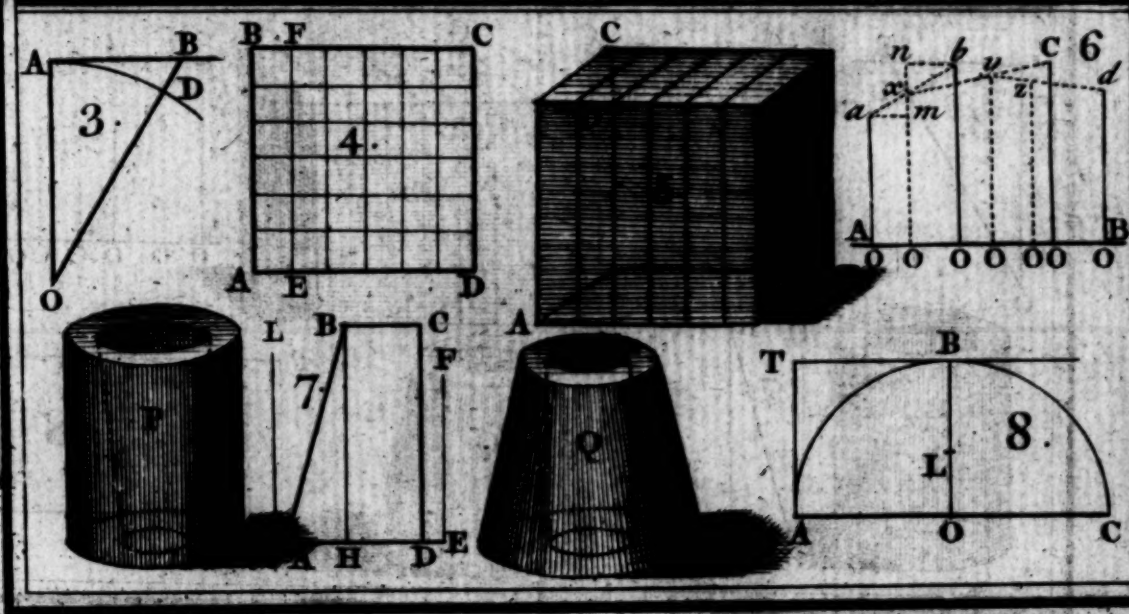
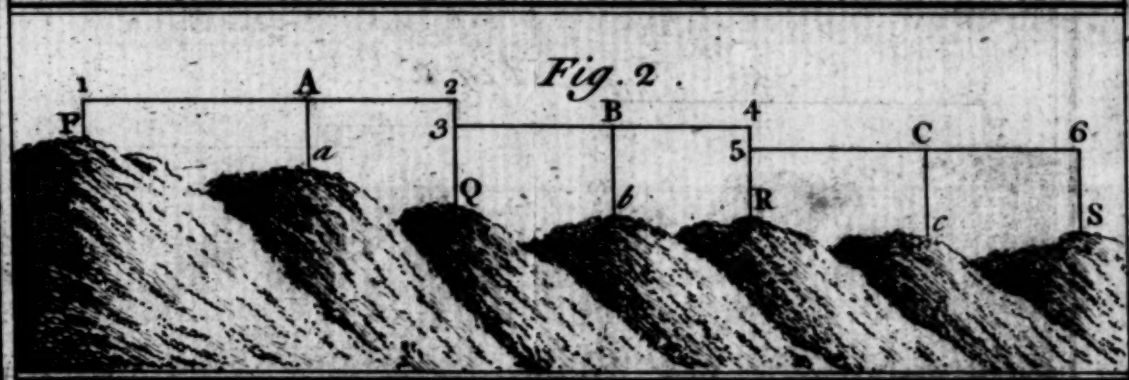


Fig. 2.



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the fig. 11; where AB and AD are each 5 feet, the angles at B and D right ones, and the angle C equal to that of the salient angle of the bastion, which has been (*art.* 399) found to be 83 degrees and 8 minutes in the exagon; and since the sides CB, CD, are equal, they will be found by trigonometry to be 5.6 feet.

The contents of the upper and lower base, are equal to the rectangles made by the unequal sides, and the mean plane to that made by the least side of the one, and the greatest of the other.

The angle L at the base being equal to the angle C, that is, of 83 degrees and 8 minutes, the side EF, 11 feet as well as its equal EN, the sides FL and LN; will be found to be, by the similarity of the triangles ACD, EFL, 12.32 feet. Therefore, having the height, the lower and upper base of this frustum, its solid content is found to be 11.42 toises. And subtracting the side CD (5.6) feet, fig. 11, from the face of the bastion, fig. 12, or from 50 toises, we shall have 49 toises 4 tenths of an inch for the part CD of the face of the bastion to the orillon: this length multiplied by the content 6 toises 4 feet of the profil, gives 327 cubic toises, for the content of the face AD.

We have found (*art.* 401) the radius with which the orillon is described to be 5.13 toises, the distance from the center of gravity of the rectangular part ABCL, fig. 10, of the profil, whose base AL is 5 feet, to the center of that arc, or to the axis of rotation, will be (*art.* 422) 28.28 feet, and the distance from the center of gravity of the triangle LCD is 32.78 feet: so that having the arc of (*art.* 401) 11.16 toises, which the orillon makes near the cordon, the arc described by the center of gravity of the rectangle ABCL will be 10.5 toises nearly, and that described by the center of gravity of the triangle LCD is 11.88 toises; and because the rectangle ABCL is $4\frac{1}{2}$ toises, the part of the orillon described by that rectangle will be 43.75 toises, and the triangle LCD being 2.5 toises, the part of the orillon described by that triangle will be 29.7 toises.

Con-

Consequently, the orillon contains 73.43 cubic fathoms or toises.

The strait part FE, fig. 12, of the wall between the orillon and retired flank, being 5 toises long by construction, is easily found, for which reason we shall omit it here.

As to the retired flank EL, its radius has been found to be (*art.* 400) 18.86 toises, and the arc 19.75 toises; the radius of the arc described by the center of gravity of the rectangle ABCL, fig. 10, will be 20.27 toises, and the arc described by that center 21.23 toises nearly; the radius of the arc described by the center of gravity of the triangle LCD is 19.36 toises; and the arc described by that center 20.27 toises. Therefore, the part of the retired flank described by the rectangle ABCL ($4\frac{1}{2}$) will be 88.46 toises; and the part described by the triangle LCD (2.5) is 50.68 toises. Consequently, the retired flank contains 139.14 toises.

The broken part LH, fig. 12, of the curtain, and the curtain itself, being strait walls, are easily found, by multiplying the profil by their lengths.

It must be observed, that in all the saliant angles the parts cut off by planes perpendicular to the walls from the inner angles are always frustums of pyramids, like that represented by the 11th fig. whose upper and lower bases are known from the given dimensions of the profil; and the parts cut off in the re-entering angles are such frustums as are represented by fig 13, from which there is cut off a small pyramid, arising from the slope of the wall; and as these parts of the wall are always regular solids, their contents may be found in the manner shewn before; the same thing may be said in regard to the counterforts, which are regular prisms.

What has been said in respect to the body of the place, may likewise be applied to all kinds of outworks; the round parts of the counterscarp are found in the same manner as the retired flanks, for which reason we shall insist no longer on this subject.

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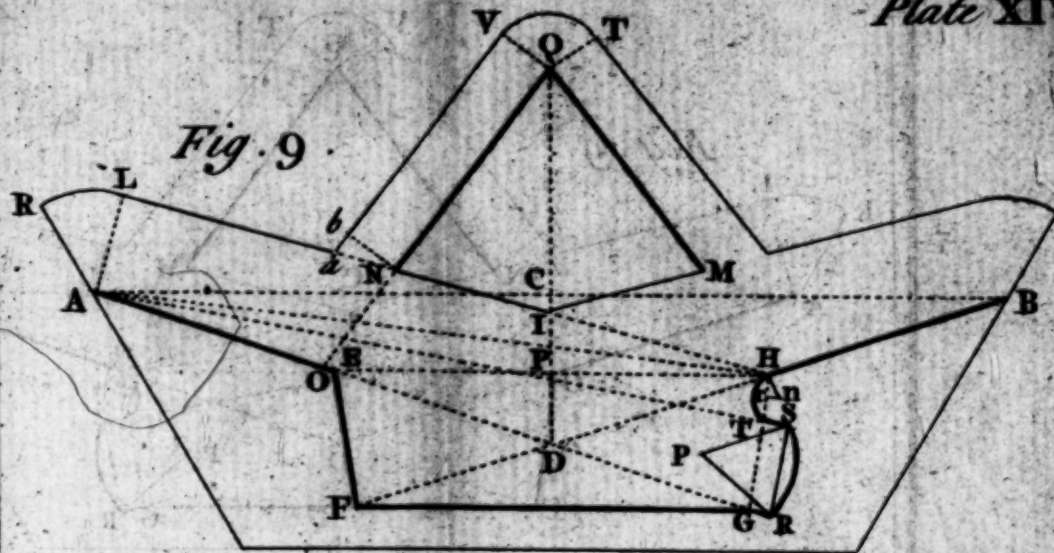


Fig. 10.

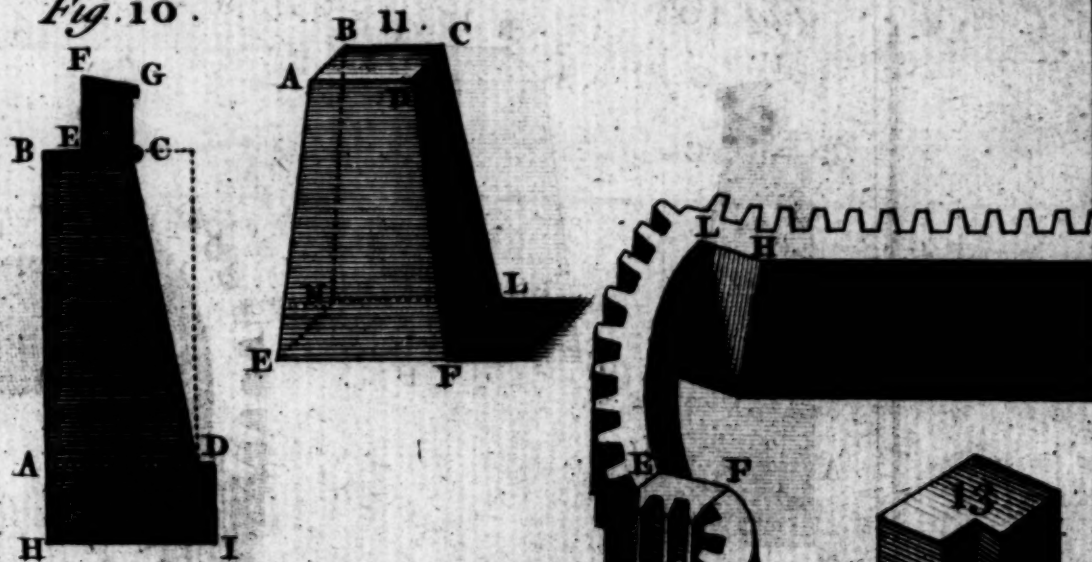
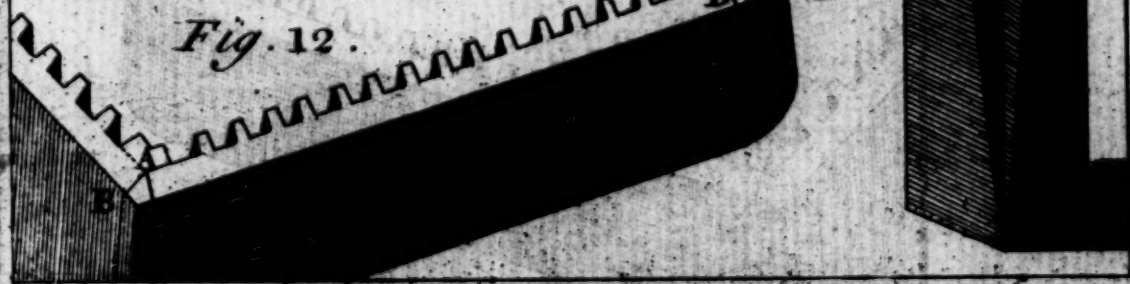


Fig. 12.



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MENSURATION applied to VAULTS and ARCHES.

There are three sorts of arches, the circular, the elliptic or oval, and the gothic; we shall begin with the mensuration of the first, as being mostly used, especially here in *England*.

The first thing of this kind to be considered in a fortification are the powder magazines; their planes are generally as is represented by fig. 14, and the section as fig. 15; the outsides of the roofs being always in the form of a right angled triangle. Plate XV. Fig. 14, 15.

The common method for measuring these arches is to consider the part EDF as a solid pyramid, whose base is the triangle EDF, and whose altitude the length of the magazine; then the content of the semi-cylinder, whose base is the semi-circle APB, and the axis the same as that of the pyramid, being subtracted from the pyramid, the difference gives the content of the upper part of the magazine, cut off by the horizontal line EF. But the work will be shorter if the content of the semi-circle APB be subtracted from that of the triangle EDF, and the difference multiplied by the length of the inside AC, fig. 14, of the magazine.

Fig. 14. The side walls are measured by multiplying the plane or base ABCD by the height LA, fig. 15, of the peers; the contents of the foundations and the two gable ends, are easily found, as being regular solids.

The Sally ports being likewise arches much like the preceding one, only the outside being circular and concentric with the inside, instead of triangular; so there is no more to do than to subtract the content of the inner semi-circle from that of the outward one, and multiplying the difference by the length of the arch; the side walls are found as before.

If the arches are elliptical, such as most gateways generally are, then because the area of the circle described with the greater axis as diameter is to the area of the ellipsis,

ellipsis, as (*art.* 316) the greater axis is to the least. These kind of arches may be measured with as much ease as if they were circular.

Fig. 16. In order to measure the gothic arches, it is necessary to know their construction: the opening AB of the arch is divided into three equal parts AO, OC, and CB; from the point C, as center, with CA the two thirds of AB as radius, the arc AL is described; and from the other point O as center, with the same radius as before, the arc BL is described.

Hence, if the line DLP bisects AB at right angles, CP being the half of one third of AB will be the sixth part of AB; and as CA or CL is two thirds of the same line, if we say as $\frac{2}{3}$ is to $\frac{1}{6}$, or as 4 is to unity, so is the radius 100000 to the cosine 25000 of the angle LCP; this angle will be 75 degrees 31 minutes; the perpendicular PL is likewise found to be .065 parts of AB: Now if the content of the sector CAL be found, and the area of the triangle CPL subtracted, the difference will give the area ALP of half the inside arch; so that if twice that difference be subtracted from the area of the triangle EDF, and the remainder multiplied by the length of the arch, you will have its solid content.

The most difficult part of measuring arches is when there are several that cross each other at right angles, and are supported by pillars, such as those under the treasury near St. James's park; and above all, when they are made in the gothic manner, as those at *Westminster*: but as there are but few or none built now in that manner, we shall only insist on the method of measuring the former.

The 17th figure represents the manner in which they are joined or meet each other; this is nothing else than two semi-cylinders intersecting each other, so that the only difficulty consists in measuring the part ABCD.

If the line FG be drawn through the point of intersection E of the arches, parallel to AB or DC, and we suppose a plane to pass through that line, perpendicular to

to

to the base $ABCD$; it is evident that this plane bisects the fourth part BEC of this arch; and therefore, having found the part BEG or GEC by multiplying it by 8, we shall have the content of the whole.

Fig. 18. Let AKC be a semi-cylinder, whose diameter AC is equal to that of the arch; and let ABC be a section, so as its altitude Bn be equal to half the diameter AC ; if DBE be a section parallel to the base AnC , it is easy to perceive that the part CBE or ABD is equal to the part BEG , fig. 17, of the arch; and therefore the two parts, CBE , ABD , together; or, which is the same, the difference between the semi-cylinder $ADBEC$, and the part ABC cut off, will be equal to the fourth part BEC , fig. 17, of the arch.

If the part ABC be cut by any two parallel planes OnB , Pmr , perpendicular to the base AnC of the cylinder, they will form two right angled similar triangles OnB , Pmr : For the lines nB , mr , being parallel and perpendicular to the base AnC of the cylinder, by supposition, will be perpendicular to the parallels On , Pm ; and since BO , rP , are likewise parallel, and in the same plane, they will make equal angles BOn , rPm , with the parallels On , Pm .

Now because similar triangles are as (*art. 88*) the squares of their homologous sides, the areas OnB , Pmr , are to each other as the squares of the lines On , Pm ; that is, as the squares of the ordinates of a semi-circle: But the squares of these lines are likewise as the areas of (*art. 170*) the circles which they describe in the rotation of the semi-circle AnC about the diameter. Therefore the part ABC will likewise be as the sphere described by the semi-circle AnC . And because the content of the sphere is found (*art. 216*) by multiplying the area of the circle described by the radius On , by the two thirds of the diameter AC ; by a parity of ratios, the solid ABC will likewise be found, by multiplying the area of the greatest triangle OnB by the two thirds of the diameter AC .

Hence, because the lines On , nB , are equal by supposition,

position, and On is the radius of the semi-circle AnC, the area of the triangle OnB will be half the square made by the radius; and this multiplied by two thirds of the diameter AC, gives the two thirds of the cube made by the radius, equal to the content of the solid ABC.

If the radius OC or OA be a , the semi-circumference AnC, c ; then will $\frac{2}{3}a^3$ express the content of the solid ABC, and (art. 235) $\frac{1}{2}aac$, that of the semi-cylinder ADEC: Therefore $\frac{1}{2}aac - \frac{2}{3}a^3$ or $\frac{1}{6}aa \times 3c - 4a$ will be the content of their difference ADBEC, or the fourth part of the arch ABCD, fig. 17. Consequently, $\frac{2}{3}aa \times 3c - 4a$ will express the content of the whole, which gives this

GENERAL RULE.

Subtract twice the diameter of the base from three times the semi-circumference, and multiply the difference by the two thirds of the square made by the radius.

If the diameter be 14 feet, the circumference will be 44; and 28 twice the diameter subtracted from 66 three times the semi-circumference 22, gives 38 for the difference; and as the square of the radius 7 is 49, the two thirds of this is $32\frac{2}{3}$, which being multiplied by 38, gives $1241\frac{1}{3}$ for the content of the solid.

The figure 18, of which we have just now given the content, is the vacant part of these kinds of arches, and therefore is to be subtracted from the whole structure, considered as all solid.

The part or solid ABC is called the Hoof or *Ongled*, and its content is necessary in the measuring of round towers placed on Batardeaux.

There is a kind of arches called by the *French* Bonnet de Pretre; the ciellings of round towers are sometimes made in this manner, and are supported by a round pillar when the building is spacious.

The vacant part of these arches is a solid described by the rotation of a semi-circle about an axis, which is perpendicular to the diameter, therefore may be measured by what has been said in article 430.

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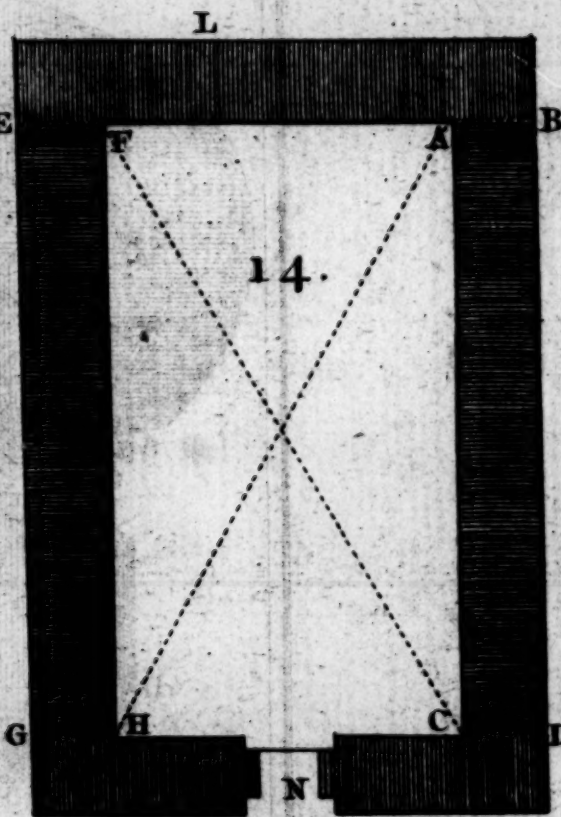
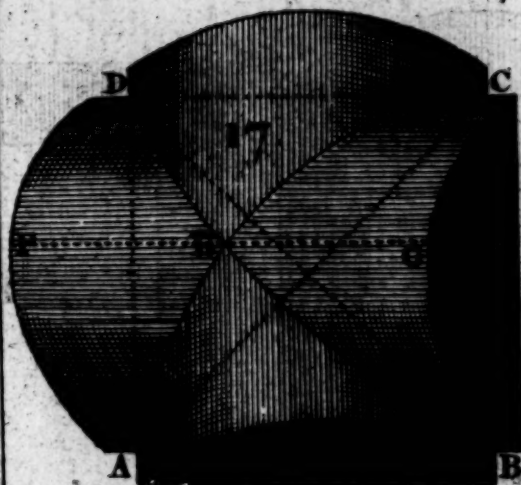
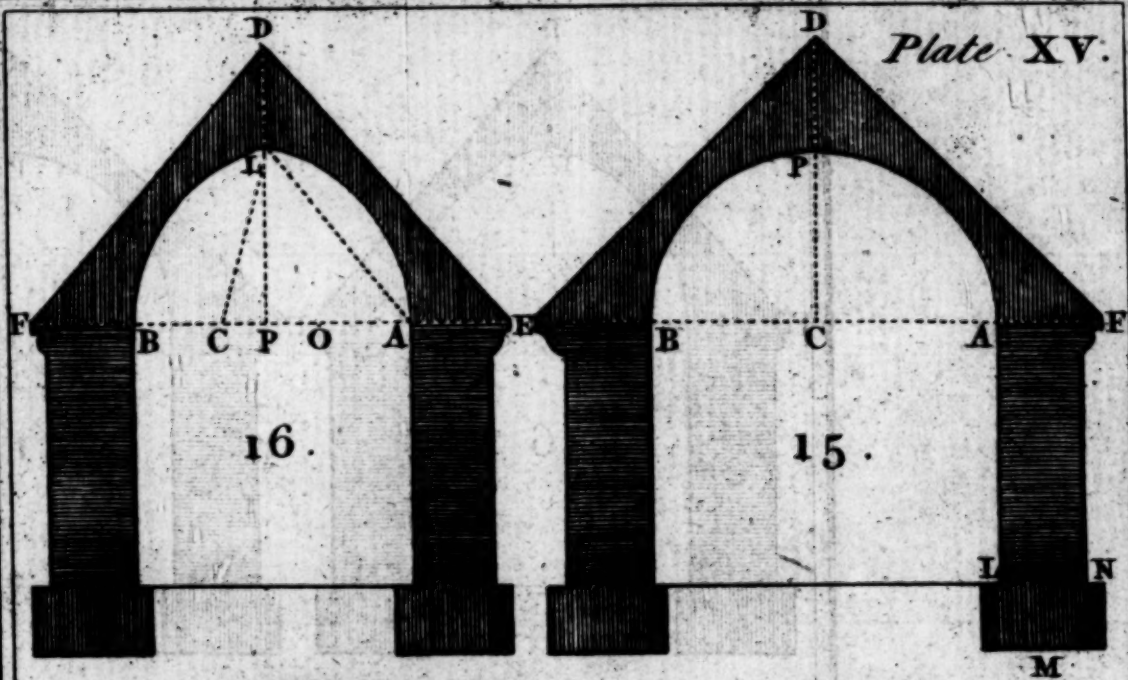
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BOOK IV.

Of MOTION.

S E C T. I.

D E F I N I T I O N S.

1. **MOTION** is a continual successive change of places.

2. Velocity is that affection of motion by which a body moves over more or less space in equal times.

3. Uniform motion is when a body moves over equal spaces in equal times.

4. Accelerated motion. when the velocity continually increases, and retarded when it continually decreases.

5. Momentum, or quantity of motion, is the power or force by which a moving body strikes any obstacle or impediment which opposes its motion.

6. Absolute motion is the change upon immoveable places, and relative, the change upon moveable places.

It seems to be plain that there is no absolute motion, for all heavenly bodies move continually in their orbits, as far as we know, and therefore the motion of bodies upon them must be relative in respect to their motion; but let this be as it will, the laws are the same in both cases, as has been confirmed by experiments; for which reason we shall treat of absolute motion only.

A X I O M.

Effects are always proportional to their adequate causes; that is, a force double produces an effect double, a triple, triple, &c.

Of UNIFORM MOTION.

T H E O R E M.

433. *Spaces described with an uniform motion, are as the times in which they are described.*

For since bodies move over equal spaces in equal times, by defin. 3, a body will move over twice the space

space in twice the time, thrice the space in thrice the time, and, in general, n times the space in n times. Therefore, spaces moved over with an uniform motion, are as the times in which they are described.

Thus a man keeping constantly the same pace, if he walks a mile in one hour, will walk two in two, three in three, and so on.

THEOREM:

434. *Spaces described with an uniform motion in equal times, are as the velocities with which they are described.*

It is plain that with a greater velocity a greater space is moved over at the same time, and that with a velocity double a space, double will be moved over at the same time; with a velocity triple, or quadruple, a space triple, or quadruple; and, in general, with a velocity n tuple, a space n tuple. Consequently, spaces moved over with an uniform motion in equal times, are as the velocities with which they are described.

THEOREM.

435. *Spaces uniformly described are in the compound ratio to that of the times and velocities.*

If two bodies A, B, describe the spaces S, s, with the uniform velocities V, v, in the times T, t, respectively: I say, that $S:s::TV:tv$.

For let another body C describe a space x , in the time T, of the body A, with the velocity v of the body B; then, because the bodies B, C, describe the spaces s, x with the same velocity v, they are (art. 433) as the times; that is, $x:s::T:t$: And since the bodies A, C, describe the spaces S, x , at the same time they are (art. 434) as the velocities V, v, viz. $S:x::V:v$. Now if the terms of these proportions are multiplied together respectively, we (art. 80) shall have $Sx:x:sx$, or (art. 95) $S:s::TV:tv$. Consequently spaces uniformly described are in the compound ratio to that of the times and velocities.

436. Hence, if two bodies A, B, describe equal spaces with an uniform motion, the velocities are reciprocally as the times. For because $S=s$ by supposition,

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we have $TV = tv$ by equality of ratios; and therefore (art. 74) $T:t::v:V$.

437. It is evident that the times are as the spaces directly, and velocities reciprocally; for if in $S:s::TV:tv$, the antecedents are divided by V , and the consequents by v , we shall (art. 54) have $\frac{S}{V}:\frac{s}{v}::T:t$. And, by the

same manner of reasoning it may be proved that the velocities are as the spaces directly and times reciprocally.

OF COMPOUND MOTION.

THEOREM.

438. *Whatever motion, or change of motion, is produced in a body, it must be proportional to, and in the direction of the force impressed.*

For since a body, by reason of its inactivity, cannot contribute to the production of its own motion, or of any change therein, it is evident that whatever motion or change of motion is generated in bodies, it must entirely proceed from the force impressed; and as effects are always proportional to their adequate causes, it must be proportional thereto, and it must likewise be directed towards the same part with the generating force.

439. Therefore, if the body whereon the impression is made was in motion before the impulse, that motion will be accelerated or retarded, according as the force impressed conspires or opposes it.

THEOREM.

440. *If a body be urged at the same time by two forces, in the directions of and proportional to the sides AB, AC, of a parallelogram, it will by this compound motion describe the diagonal AD. Plate XVI. Fig. 1.*

For while the force impressed in the direction AB causes the body to move from A towards B, the force impressed in the direction AC will cause the body to move from A towards C; but the space moved over in the direction AB is to the space moved over in the direction AC, as the force impressed in AB to the (art. 438)

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force impressed in AC, or as AB to AC or BD, by supposition. If therefore, from any point E in AB, a line EF be drawn parallel to BD, meeting the diagonal AD in F; then because (*art. 67*) $AB:BD::AE:EF$; and since the space moved over in AB, by the force impressed in that direction, is to the space moved over in AC, by the force impressed in that direction, as AB to BD; it is evident, that if AE represents the space moved over in that direction, EF will express the space moved over in the direction AC at the same time. Consequently, the body will be in the diagonal AD; and as this proportion always subsists, whether AE is great or little, the body has moved in the diagonal AD from the beginning A, and never was out of it.

This proportion is of the utmost importance in mechanics, and in natural philosophy; for it shews how to estimate the joint effect of several powers acting together on the same object, and in what manner any given number of powers may be applied, so as to produce a desired effect.

441. Hence, the body moves over the diagonal AD, by the compound motion in the same time that it would move over the side AB or AC, by the single force impressed in those directions. For we have proved, that while the body moved from A to E by the force impressed in that direction, it moved from A to F in the diagonal by the joint forces; and therefore, while the body moved from A to B by the force impressed in that direction, it moves from A to D in the diagonal by the joint forces.

442. The force impressed in the direction of the diagonal AD, by the joint forces, is to the single force impressed in the direction AB, or AC, as AD is to AB or BD: for because the effects are proportional to their adequate causes, the force impressed in the direction AD, would cause the body to move over the diagonal AD in the same time that the force in the direction AB would cause the body to move over AB; and since the joint

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joint forces in the directions AB, AC, move likewise the body over the diagonal in the same time that the force impressed in the direction AB would move it over the side AB, it follows, that a single force, expressed by the diagonal AD, produces the same effect as the joint forces expressed by the sides AB, AC, of the parallelogram.

443. As the angle CAB, made by the directions of the two forces, decreases, so does the effect arising from the joint impression increase; because the diagonal AD, which expresses their effect, increases as that angle decreases: and therefore, when the angle CAB vanishes, or the sides AC, AB, coincide with the diagonal, the effect will then be the greatest of all that these joint forces can produce.

444. On the contrary, as the angle CAB, made by the directions of the two forces, increases, the effect of their joint endeavours decreases; and when the sides CA, AB, become one continued right line, the effect of these joint forces is the least of all; which gives the two following cases.

I. If the sides AB, AC, are equal, the diagonal AD vanishes, when these sides become one continued right line; and therefore the two forces being equal, and acting in opposite directions, destroy each other.

II. If the side AC is less than the side AB, the diagonal AD coincides with the greater side AB, when these sides become one continued right line, and is equal to the difference between the sides, AB and BD, or AC. Consequently, the body will move in the direction AB of the greater force, with a force impressed equal to the difference between the forces expressed by AB and AC.

Fig. 2. 445. If from the opposite angles B, C, the lines BE, CF, are drawn perpendicular to the diagonal AD, the force impressed by AB in the direction of the diagonal AD, will be expressed by AE, and the force impressed by AC in the diagonal by AF. For if the parallelograms EH, FG, are compleated, the forces expressed by the sides AE, AH, or EB, will produce

the (*art.* 440) same effect in the direction AB as the force expressed by that line; and the forces expressed by the sides AF, AG, or FC, will likewise produce the same effect as the force expressed by the diagonal AC; and because DC, AB, are parallel and equal, the right angled triangles AEB, DFC, are (*art.* 25) equal in all respects; so that BE and FC, or HA and AG, are equal: and since the forces expressed by HA, AG, are opposite and equal, they will destroy each other: and consequently, AE, AF, express the parts of the forces AB, AC, which act on the same body in the direction of the diagonal; and since AE is equal to DF, the sum of these parts AF, AE, is equal to the diagonal. Which confirms, that a force expressed by and acting in the direction of the diagonal, is equivalent to the joint forces acting and expressed by the two sides AB, AC.

Fig. 3. 446. Lastly, If a force AB acts upon a body, and it is required to find the part of that force which acts upon this body in any other direction DC; by drawing AD perpendicular to the direction DC, the absolute force will be to the part relative to the direction DB, as AB is to DB, or as the radius to the cosine of the angle ABD, made by these directions.

SCHOLIUM.

Fig. 1. It may easily be proved from what has been said, that relative motion observes the same laws as absolute motion; for if we suppose that the figure ABCD moves with an uniform motion in the direction AB, and a force acts upon a body in the direction AC; it is plain that whilst the line AC moves from A to E, or B, the body will by the impressed force move from E to F, or to D, in the line AC: so that a spectator standing at A will see the body move in a right line AC; for when the line AC comes into the position BD, the spectator will be at B, and still see the body move in the same direction, although the body moves in reality over the diagonal AD; and hence, whether our earth be in motion or at rest, the motions of bodies will appear to us always

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in the same manner, because we move the same with the rest of the objects placed on the surface.

Of MOTION, uniformly accelerated or retarded.

Notwithstanding that gravity is different at different heights from the surface of the earth, and farther or nearer the equator, tho' equally distant from the surface; nevertheless, the difference is so little at a small distance from the same point of the surface, that it produces no sensible error in computation; we shall therefore suppose with *Galileus* and others, that gravity acts uniformly and constantly the same in every part of time.

T H E O R E M.

447. *If a body begins to fall freely from rest, it will acquire velocities proportional to the times, from the beginning of the fall.*

Suppose the time to be divided into very small equal particles or instants; then as in each particle the force of gravity makes equal impressions on the body by supposition, whatever force therefore the body receives from the impression of gravity in the first instant, it must receive as much in every other instant; and since the impressions remain, the velocity acquired in the first instant will be doubled in the second, tripled in the third, and quadrupled in the fourth, and so on continually through every instant of the fall. Therefore, setting aside all resistance, the velocity of a falling body, freely from rest, will constantly increase in the same proportion as the times of its descent from the beginning of the fall.

448. Fig. 4. Hence, if the line AB expresses the time of the fall, BC perpendicular to AB, the velocity acquired during that time, by drawing AC, and from any point D, in AB, DE parallel to BC; then will DE express the velocity acquired in the time AD. For because the velocities acquired are as (*art. 447*) the times elapsed from the beginning of the fall; and by reason of the similar triangles ADE, ABC, we have (*art. 69*)

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$AB:AD::BC:DE$. It follows that DE expresses the velocity acquired in the time AD .

T H E O R E M.

449. *The spaces moved over by a body falling freely from a state of rest by the force of gravity, are as the times and velocities conjointly.*

Let the line AB be divided into equal parts indefinitely small, and from each of those divisions suppose lines, as DE , drawn parallel to BC : it is evident, from what has been said, that those lines express the velocities acquired in their respective times; and as the velocities acquired in very small particles of time may be considered as uniform, the spaces described in these small particles of time (*art. 435*) will therefore be as the times and velocities: but the sum of the spaces described in all the small portions of time is equal to the space described from the beginning of the fall; and as the sum of all the rectangles of the lines, as DE , and the small parts of AD , are equal to the triangle BDE ; it follows, that the space described in the time AD is to the space described in the time AB , as the triangle ADE is to the triangle ABC ; that is, as the rectangle of AD and DE to (*art. 91*) the rectangle of AB and BC .

450. Since the similar triangles ADE , ABC , are to each other as (*art. 88*) the squares of their homologous sides AD , AB , or DE , BC ; it is evident that the spaces described by a falling body, from a state of rest, are as the squares of the times elapsed from the beginning of the fall, or as the squares of the velocities acquired during the fall.

451. If the rectangle $ABCF$ be compleated, then the space described uniformly with the velocity AF or BC , in the time AB , will be (*art. 435*) as the rectangle $ABCF$; and the space described by the accelerated motion of a falling body, in the same time AB , as the triangle ABC ; and since the rectangle BF is (*art. 35*) double the triangle ABC , it follows, that the space described uniformly with a velocity acquired by a falling body

body from a state of rest, is double the space described in the same time by the falling body.

R E M A R K.

It has been found by experiments, that a body falling in the latitude of *London* from a state of rest, describes a space of 16.1 feet, *English* measure, in a second of time; this being supposed, by what has been said above, we may find the space a falling body will describe in any other time. For example, the space described in 4 seconds, will be found by saying, as the square 1 of 1 second is to the square 16 of 4 seconds, so is the space 16.1 feet described in 1 second to the space 257.6 feet described in 4 seconds.

On the contrary, we find the time required by a falling body so as to describe any space 144.9, by saying as 16.1 feet is to 144.9 feet, so is the square of one second to the square of the time required, which will be 9, whose square root 3 will be the time required.

T H E O R E M.

452. *If a body be thrown directly upward, the time of its rise will be equal to the time AB, wherein a body falling freely from a state of rest acquires the same velocity BC, wherewith the body is thrown up.*

For since the action of gravity is constant and uniform in whatever time it generates any velocity in a falling body, in the same time, in a rising body, by reason of its acting in a contrary direction, it must destroy that velocity: therefore, if the time AB of a falling body be divided into equal particles or instants, whatever velocity is acquired in each part of time in the falling body, the same velocity is destroyed in the raising body at the same time: and consequently, at the end of the time AB, the whole velocity BC of the raising body will be destroyed.

T H E O R E M.

453. *The height to which a body thrown directly upward rises, is equal to the height from which a body falling freely from a state of rest, acquires a velocity equal to that wherewith a body is thrown upward.*

For since the times in which the velocity of a falling body is generated, and that of a raising body destroyed, are equal; and because of the two equal velocities, one is generated and the other destroyed, by the constant uniform action of gravity: It is manifest, that whatever the space through which the falling body moves, in order to acquire its velocity, the rising body must ascend through an equal space in order to lose its velocity; that is, it must rise to the same height from which the other falls.

T H E O R E M.

454. *All bodies falling from a state of rest, through equal spaces, acquire by the force of gravity equal velocities; that is, two bodies of different weights acquire the same velocity in the same time, setting aside all external obstacles and impediments.*

Suppose two bodies of the same kind, one of which being double in weight to the other, if they are let fall from the same height, at the same time, they will both together arrive at the same instant to the ground; for the greater body may be divided into two bodies, each equal to the lesser, which being let fall together will describe equal spaces in the same time; but whether this body be separated into equal parts or not, it is evident that the force of gravity acts the same; therefore a body, double another body, falls with the same velocity.

If one body is to the other in any proportion, the greater may be always divided into a certain number of equal bodies which are equal to the least; and if these parts are let fall together, they will pass over equal spaces at the same time; but as the gravity of the parts is equal to the gravity of the whole, the forces in the parts arising from the action of gravity, must be equal to the force of gravity in the whole. Consequently, all bodies accelerate equally at the same time.

It is also confirmed by experiment, that all bodies, light or heavy, fall equally swift, in an exhausted receiver, as gold and a feather: So that if there is any difference

difference in the open air, it is owing to the resistance of the medium in which they move.

Of the COLLISION of hard non-elastic bodies.

By hard non-elastic bodies we mean, those which, if they strike any obstacle, do not change their figure, and are void of all springs or elasticity.

T H E O R E M.

455. *Reaction is always equal to action; that is, the actions of two bodies upon one another are always equal and in contrary directions.*

For whatever change is made in the state of one body, whether at rest or in motion, by the action of another, the same change is produced in the state of the other by the reaction of the former; and the directions of those changes are contrary to each other.

T H E O R E M.

456. *The momentums or forces with which two unequal bodies A and B strike another body C, either in motion or at rest, with the same velocity, are as the weights of those bodies.*

If the body A be double the body B, it is plain that it may be divided into two bodies, each equal to B; and these two bodies having the same velocity as B, would each strike the body C with the same force as the body B; but whether the body A be separated into parts, or united in one, it will strike the body C with the same momentum. If the body A be triple the body B, it will, by the same reason, strike the body C with a force triple to that of the body B; and therefore, whatever the proportion is between the bodies A and B, they will, because the effects are proportional to their adequate causes, strike the body C with forces that are in the same proportion.

T H E O R E M.

457. *If two equal bodies, A and B, strike another body C, either in motion or at rest, with unequal velocities, their momentums will be as the velocities.*

For

For if the velocity of the body A be double the velocity of the body B, it will strike the body C with twice the force to that of the body B; if its velocity is triple, with a force triple; if quadruple, with a force quadruple; and so on, in the same proportion of the velocities. For since the effects are proportional to their adequate causes, the bodies being the same, the effects or forces must be proportional to the causes or velocities.

T H E O R E M.

458. *If two unequal bodies, A and B, strike another body C, either in motion or at rest, with unequal velocities; their momentums will be as their weights and velocities conjointly.*

For if W, w , express the weights of the bodies A and B; V, v , their velocities; and M, m , their momentums respectively; let there be another body D, whose weight is W , velocity v , and momentum x ; then because the bodies A and D, having the same weight W , their momentums M, x , are (*art. 457*) as the velocities V, v ; that is, $M : x :: V : v$; and since the bodies B, D, have the same velocity v , their momentums m, x , are as (*art. 456*) their weights w, W ; viz. $x : m :: W : w$. If the terms of these proportions are multiplied together, we shall have $Mx : mx$, or (*art. 65*) $M : m :: WV : wv$. Consequently, the momentums of two unequal bodies A, B, striking another body C, either in motion or at rest, with unequal velocities, are as their weights and velocities conjointly.

459. Hence, if the momentums M, m , are equal, we have $WV = wv$ by equality of ratios, or $W : w :: v : V$. Consequently, when the momentums are equal, the weights are reciprocally as their velocities; and, on the contrary, if the weights are reciprocally as the velocities, the momentums are equal.

460. The weights are as the momentums directly, and the velocities inversely: For if the antecedents in $M : m :: WV : wv$, are divided by V , and the consequents by v , we shall have (*art. 54*) $\frac{M}{V} : \frac{m}{v} :: W : w$.

And

And by the same way of reasoning the velocities are as the momentums directly, and the weights inversely.

T H E O R E M.

461. *If a non-elastic hard body A strikes another body B of the same kind, either in motion or at rest; the sum or difference of their momentums will remain the same after the stroke, as it was before, and they will move together in the direction of the body which had the greatest motion before the stroke.*

For since reaction is always equal to (art. 455) action, and in a contrary direction when the bodies move the same way, the one loses as much of its motion as it communicates to the other; and when they move contrary ways, the body which had the greatest motion, will lose as much of it as the other had before the stroke, and they will continue to move after the stroke in the direction of the body which had the greatest motion before the stroke. If the body B is at rest, the body A will lose as much of its motion as it communicates to B: Therefore they will continue to move with the sum of their momentums, if they went the same way, or with their difference if the contrary ways, and in the direction of the body which had the greatest motion before the stroke.

462. Hence, because the bodies A and B move together after the stroke, whether they moved before the same or contrary ways, or the body B was at rest; it is manifest, that they will both have the same velocity after the stroke.

463. Since the momentums are in the compound (art. 458) ratios of the weights and velocities, and the bodies have the same velocity after the stroke; it is plain that the sum of the momentums before the stroke divided by the sum of the weights, will give their common velocity after the stroke, if they moved the same way; and the difference between their momentums divided by the sum of their weights, if they moved contrary ways; but if B was at rest before the stroke, the momentum

mentum of A, divided by the sum of the weights, will give their velocity after the stroke.

If A and B express their weights, a , b , their respective velocities before the stroke, and v their common velocities after the stroke; then will

$$v = \frac{aA + bB}{A + B}, \text{ when they move the same way.}$$

$$v = \frac{aA - bB}{A + B}, \text{ when they move contrary ways.}$$

$$v = \frac{aA}{A + B}, \text{ when B is at rest before the stroke.}$$

Examp. Let the weights A and B be as 3 to 1, and their velocities a , b , as 6 to 2; then will 18, 2, be as their momentums, and 4 the sum of their weights: Therefore $v = \frac{18 + 2}{4} = 5$, in the first case; $v = \frac{18 - 2}{4} = 4$,

in the second, and $v = \frac{18}{4} = 4.5$, in the third.

464. Since the body A loses as much of its motion by the stroke as B gains by it; and when two quantities of motions are equal, the weights are (*art.* 459) reciprocally as the velocities; it follows, that if x be the velocity lost by A, and y that gained by B, we have $A : B :: y : x$, and by composition $A + B : B :: y + x : x$; or $A + B : A :: y + x : y$. Consequently, the sum of the bodies is to either, as the sum of the velocities gained and lost is to the velocity of the other body.

465. It is evident, that the body A strikes the body B with the difference $a - b$ between their velocities, when they move the same way; or with their sum $a + b$, when they move contrary ways; and only with its own when B is at rest; and as it is this velocity which is divided between the bodies A and B after the stroke, $x + y$ is equal to $a - b$ in the first case, to $a + b$ in the second, and to a in the third. Consequently, $A + B : B :: a - b : x$, and $A + B : A :: a + b : y$, in both cases, and b is nothing in the last.

The same velocities are also found by taking the differences

ferences between the common velocity v after the stroke, and the velocities a and b , which they had before the stroke.

Of the COLLISION of ELASTIC BODIES.

Bodies are said to be Elastic, if they consist of such parts as yield and give way when pressed, and restore themselves upon the removal of the pressure.

If the restoring force is equal to the force of compression, then are these bodies said to be perfectly Elastic; but if they are unequal, the ratio of these forces is called the Elastic Ratio.

It is to be observed that all homogeneous bodies of the same kind, have their elastic ratio constant and invariable: For whatever the cause of elasticity may be, it must be the same in all bodies whose constituent parts are the same.

T H E O R E M.

466. *If two elastic bodies A, B, move the same way, or the body B be at rest, the sum of their momentums after the stroke will be always equal to the sum of their momentums before the stroke.*

For because reaction is always equal to action, and in a contrary direction, if the bodies move the same way, whatever motion A loses by striking B, the same motion will be gained by B in the stroke, whence the sum of their motion will remain the same in that instant; and whatever the force is, by which the parts restore themselves to their former state, it must act in the same manner on both; therefore, the body A loses as much by the restoring force, as B gains by it. Consequently, the sum of their momentums after the stroke, is equal to the sum of their momentums before the stroke.

467. Since the quantity of motion lost by the body A, by the force of compression, is equal to the quantity gained by B; it is evident, that the body A is to the body B, as the velocity gained by B is to (art. 459) the

the velocity lost by A; and therefore, by composition, (art. 75) $A+B:B::a-b:x$, and $A+B:A::a-b:y$, 468. It is evident, that the velocities lost and gained consist of two parts, the one arising from the force of compression, and the other from that with which the parts restore themselves, which parts are equal in perfect elastic bodies, and in a given ratio in non-perfect elastic bodies.

If a' , b' , express the velocities of A and B after the stroke, and unity is to r , as the compressing force is to the restoring force; then because $A+B:B::a-b:$
 $\frac{B \times a - b}{A+B}$ = to the velocity of A, lost by the compressing force, and $A+B:A::a-b:$
 $\frac{A \times a - b}{A+B}$ to the velocity of B, gained by that force, their respective forces, lost and gained by the restoring force, will be $\frac{r B \times a - b}{A+B}$, and $\frac{r A \times a - b}{A+B}$; adding the two forces lost by A, we shall get $\frac{1+r \times B \times a - b}{A+B}$, for its velocity lost, and $\frac{1+r \times A \times a - b}{A+B}$, for the total gain by B. Consequently,

$$a' = a - \frac{1+r \times B \times a - b}{A+B}$$

$$b' = b + \frac{1+r \times B \times a - b}{A+B}$$

If the bodies are perfectly elastic, then will $r=1$; and therefore.

$$a' = a - \frac{2B \times a - b}{A+B}$$

$$b' = b + \frac{2A \times a - b}{A+B}$$

And

And if the bodies are void of elasticity, we shall find the same expression for both velocities as before; therefore this theorem is general, and extends to all cases that can happen in the collision of elastic, or non elastic bodies: When the bodies move contrary ways, we need but change the sign of b , into $+$ when $-$, or into $-$ if $+$, in order to have the different expressions for that case.

469. If the perfect elastic bodies A and B are equal, and the body B was at rest before the stroke; by making $A=B$, and $b=0$, we shall have $a'=0$, and $b'=a$; which shews, that the striking body A remains at rest after the stroke, and the body B flies off with the velocity a of the former.

470. If the perfect elastic bodies are equal, and move contrary ways, by making $A=B$, and changing the signs of b , we get $a'=-b$, and $b'=a$; which shews that both bodies reflect backwards and change their velocities.

471. If the body A strikes the body B at rest, with the velocity a , and the body B strikes another body C at rest, with the velocity b' ; then if c' expresses the velocity acquired by C, we shall have $c' = \frac{1+rBb'}{B+C}$: or because $b' = \frac{1+rAa}{A+B}$; there will be $c' = \frac{1+r^2 \times ABa}{A+B \times B+C}$.

S E C T II.

Of M E C H A N I C S.

D E F I N I T I O N S.

1. **M**ECHANICS is the science which considers the proportions between powers that act upon bodies, and the effects they produce when applied to machines. Fig. 18.

2. Two or more powers are said to be in Equilibrio when their mutual actions, on the same object, destroy each other so as to remain at rest.

3. Static

3. Static is that part of mechanics which considers the proportions of powers in a state of equilibrio.

4. The direction of a power is the line in which it acts upon the body, or compels it to move in.

5. All kinds of instruments which serve to move bodies, or to stop their motions, are called Engines, or Machines; there are several sorts, simple and compound.

6. The simple are five, viz. the Lever, the Pulley, the Wheel, the Inclined Plane or Wedge, and the Screw.

The compound machines are without number, but are all made by the combination of the five simple ones.

7. The absolute force of any power, is that which, when applied either to move or to stop bodies in motion, suffers no diminution from the direction in which it acts.

8. The relative force of a power is that part of the absolute force, which serves to produce the desired effect, while the other part is destroyed by some opposition.

A X I O M.

Gravity acts constantly the same, within a small distance of the surface of the earth, and in directions parallel to each other.

Though this is not true, according to a geometrical exactness, yet it answers so nearly, that by the nicest computations we can make, there is no sensible difference to be found; for which reason *Galileus*, and other authors, admit this axiom as an undeniable truth.

T H E O R E M.

472. *If two powers P, Q, sustain a weight W, by means of strings, so as to remain at rest, they will be to each other as the sides of the parallelogram made of their directions, that of the weight being in the diagonal.* Plate XVI. Fig. 5.

For let ADBC be the parallelogram, described on the directions CB, CA, of the powers, and CD that of the weight; then because the forces expressed by and in the directions of the two sides CA, CB, are (*art.* 442) equivalent to the force expressed by and in the direction of the

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the diagonal; and the weight W acts downwards from D to C , while the powers P , Q , act from C to A and B ; the action of the weight being equivalent to the joint actions of the powers, and in a contrary direction, they must destroy each other, and therefore remain in a state of rest. Consequently, these powers P , Q , are to each other as the sides CA , CB , of the parallelogram.

473. Hence, as the angle ACB made by the directions of the powers P , Q , opens, so these powers must increase in order to sustain the weight W , which we suppose to remain the same; and when the directions CA , CB , become one continued right line, it is evident that the powers P and Q must be infinite to support the least weight W possible; which shews that no finite power is able to stretch a rope of any length, so as to make it strait.

T H E O R E M.

474. *If three powers P , Q , R , fixed to an inflexible line AB , by means of strings, are in equilibrio; the directions of the two powers Q , R , on the same side of this line, will, when produced, meet the direction of the other power P in the same point D , if they are not parallel. Fig. 6.*

From the point C , where the power P is fixed to the inflexible line, draw CE and CF parallel to the directions AD , BD , of the powers Q , R ; then as it is the same thing whether these powers are fixed to the line AB , or to any other points in their directions, they act in the same manner as if they were fastened together by means of strings, AD , BD , and CD ; and therefore they will be to each other (*art.* 472) as the sides CE , CF , of the parallelogram made of their directions, since they are in equilibrio by supposition. But this cannot happen unless the directions AD , BD , of the powers Q and R meet the direction CD of the opposite power P , in the same point D . Consequently, their directions must meet in the same point when produced.

Q

475.

475. Since (*art.* 472) $R : Q :: ED : FD$; or because $CF = ED$, and $CE = FD$, this proportion will be $R : Q :: CF : CE$. Consequently, two powers Q and R are in equilibrio with a third P , when they are reciprocally as the lines CE , CF , drawn from the point, fix C parallel to their directions, and terminated by the same.

476. If from the point C the lines CL , CK , are drawn perpendicular to the directions BD , AD ; then because CF , LE , and CE , KF , are parallel by supposition, the angles CFK , CEL , formed by these parallel lines, are equal; and therefore the right angled triangles CKF , CLE , are (*art.* 67) similar; whence $CF : CE :: CK : CL$. And since (*art.* 475) $R : Q :: CF : CE$, we have $R : Q :: CK : CL$, (*art.* 53) by equality of ratios. Consequently, the powers Q , R , are also reciprocally as the perpendiculars drawn from the point C to their directions, when they are in equilibrio.

477. If any two directions of the three powers which are in equilibrio are parallel to each other, the third must likewise be parallel to these two. For should AD and BD be parallel, and CD meet any of the two former, it is evident that these powers (*art.* 474) could not be in equilibrio, which is contrary to supposition.

478. Hence, if the directions of three powers which are in equilibrio, are parallel, the two powers Q , R , on the same side of the inflexible line AB , will be reciprocally as the parts of that line, terminated by the point C , and their directions.

Fig. 7. For if the directions are perpendicular to the line AB , the perpendiculars CK , CL , in fig. 6, will coincide with the parts CA , CB ; and therefore the proportion (*art.* 476) $R : Q :: CK : CL$ becomes $R : Q :: CA : CB$.

Fig. 8. When the directions are oblique to the line AB , these perpendiculars CK , CL , become one continued right line; and since the right angled similar triangles CKA , CLB , give (*art.* 69) $CK : CL :: CA : CB$; and because (*art.* 476) $R : Q :: CK : CL$, we have $R : Q :: CA : CB$, by equality of ratios.

N. B. This

N. B. Tho' the *art* 478, which is of the greatest use in mechanics, has been proved in *art*. 420, we shall repeat it again in the following

T H E O R E M.

479. If two weights Q, R , are suspended by an inflexible line AB , which is supported at the point fix C , they will be in equilibrio, if they are reciprocally as the distances CA, CB , of their directions from the point fix C .

Produce the line AB both ways, so as AE be equal to BC , and BF to AC : take BD equal to BF or to AC ; then will AD be equal to AE or BC , and DE double the line BC , as likewise DF double the line AC : Now if the weights Q, R , are reciprocally as AC, BC , we have $Q : R :: (2BC : 2AC ::) DE : DF$. Hence, if DE expresses the weight Q , DF will express the weight R ; and because the lines DE, DF , are bisected in A and B , if they were suspended in those points, there is no reason for their inclining more one way than another; and since EA is equal to CB , and AC to BF , by construction, the line EF is bisected in C ; and therefore, if it be suspended by the point C , it will also remain in equilibrio. Consequently, if instead of DE the weight Q be suspended to the point A which bisects it; and instead of DF the weight R be suspended at the point B ; then because $Q : R :: DE : DF$, these weights will have the same power to move the line AB , as the lines DE and DF , which are proportional to them; but the line EF will be in equilibrio, if suspended by the point C , which bisects it. Consequently, the powers Q and R , which are reciprocally as the distances CA, CB , of their directions from the point C , will also be in equilibrio.

Fig. 7. 480. Since we have always $R : Q :: CA : CB$, when the powers or weights Q and R are in equilibrio, and (*art*. 72) $CA \times Q = CB \times R$; it follows, that the momentums of powers suspended by an inflexible line are in the compound ratio of these powers and the distances of their directions from the point of suspension C .

Fig. 8. 481. If the inflexible line AB moves about the point fix C: the vertical ascent of one power will be to the descent of the other, reciprocally as these powers: For AK expresses the ascent of the power Q, and BL the descent of the power R; and because (*art.* 476) $R : Q :: CK : CL$, when these powers are in equilibrio, we have by the similarity of triangles $CK : CL :: AK : BL$. Consequently, $R : Q :: AK : BL$.

It is from this principle that *Descartes* has demonstrated his mechanics: But tho' it has been proved before, that if the momentums of striking bodies are equal, and in contrary directions, their motions will be destroyed; yet it does not appear clearly from thence, that bodies or powers fixed together, which are reciprocally as the distances of their directions from the point fix, should be in equilibrio; for which reason we have demonstrated the principles of mechanics in the usual manner, and deduced this by way of corollaries.

482. Since $R : Q :: AC : CB$, we have by composition $R + Q : R :: AB : AC$; that is, the sum of the weights is to any one of them, as the length of the line AB is to the distance of the direction of the other from the point of suspension C. Consequently, the weights and the length of the line AB being given, the point of suspension may be found.

Examp. Let the weight Q be 8 pounds, and the weight R, 6; and let the line AB be 12 feet; then the proportion above gives $14 : 6 :: 12 : AC = 5\frac{1}{2}$. If one of the weights Q, be 20 pounds, and the distances CA, CB, are 4 and 5 feet, respectively; the other weight R will be found by this proportion, $BC (5) : CA (4) :: Q (20) : R = 16$ pounds.

R E M A R K.

It is upon this principle that the common scales for weighing all sorts of commodities are made; for when the weights Q and R are equal, the distances CA, CB, are also equal; but as it often happens that one arm is made longer than the other, and the weight of the other

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side is somewhat greater, in order to make the scale hang even, by which the buyer is imposed upon; this may be easily known; for when the goods are weighed, if you put them on the other side where the weight was, and the weight in their place, then if the scale is not true you will easily discover the cheat.

Fig. 10. The *Roman* balance, or steel-yard, is likewise made upon the foregoing principles; its construction is as follows: Take an iron rod *FB*, somewhat tapered at one end *F*, and suspend it at a point *C*, pretty near the heavier end *B*, suspend any weight *R* to the end *B*, and any other *P* at *A*, so as to slide along the rod; move this weight backwards and forwards till it balances the weight *R*, and mark the rod at the point of supposition with the number 1; then suspend another weight at *R*, double the former, and slide the weight *P* till it balances the other, and mark the rod with the number 2; continue thus to suspend successively such weights at *R*, as are to one another as the natural numbers 1, 2, 3, 4, &c. and mark every point of suspension of the weight *P*, with these numbers, 1, 2, 3, 4, &c. and you will have the steel-yard required.

If the first weight suspended at *R* be 1, 2, 3, 4, 5 pounds, the weight *P* suspended at the point marked 1, will also weigh one pound; and when suspended at the numbers 2, 3, 4, it will balance a weight, 2, 3, 4 times the former; by which one single weight *P* will serve to weigh from a small to a considerable weight.

THEOREM.

Fig. 11. 483. *If several weights, P, Q, R, S, are fixed to an inflexible line, in such a manner that the sum of the momentums of all the weights on one side of the point fixt, is equal to the sum of the momentums of all those on the other; they will be in equilibrio.*

For we have proved (*art.* 480) that the momentums of weights or powers, were in the compound ratios of these weights or powers, and the distances of their directions

rections from the point of suspension : If therefore the sum of the momentums of the weights P and Q, on one side of the point C, is equal to the sum of the momentums of the weights R, S, on the other, they will remain in equilibrio, since there is no reason for their moving one way or the other.

484. Hence, if there be a rod A B of iron, or of any other metal, suspended by the point C, and a weight P fixed at one end, we may find the weight S, which being suspended at the other end B, shall, together with the rod, be in equilibrio : For if the points D and E are the centers of gravity of the parts A C, C B, of the rod ; and Q, R, express the weights of these parts ; then by taking the momentums of the weights P, Q, and subtracting from their sum the momentum of the weight R, the difference divided by the distance C B will give the weight required.

T H E O R E M.

485. *If the point of suspension C be either above or below the line A B, and the weights P and Q are in equilibrio, when the line A B is in an horizontal position ; they will not be so in any other.* Fig. 12, 13.

Let C D be drawn perpendicular to A B, then will $P : Q :: DB : AD$, when these weights are in equilibrio ; but if the line A B is moved into any other position, such as *ab*, and C D is represented by the line *Cd*, draw *dH* perpendicular to A B. Now if we suppose the weights to be reunited into the point D, that is, into their common center of gravity, when A B is in an horizontal position, they will remain at rest ; but when that line inclines, the direction of these weights will be in *dH*, which being oblique to *Cd*, will make the weight Q descend, when the point C is below the line A B, as low as possibly it can, and it never returns again into the former position ; and when the point C is above the line, Q will descend and ascend alternately till such time that the line A B remains in an horizontal position.

N. B.

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N. B. In common scales the point of suspension *C* is generally a little above the beam; because when the weights are equal, they will soon come to an equilibrium; but as a small difference between the weights is imperceptible, that point should be placed exactly in the beam, in scales used for weighing silver, gold, and other valuable things.

And in draw-bridges, the point *C* is generally placed below the axis, about which it turns, on account of their being raised with a less force; but when it comes to a certain height, its motion accelerates to such a degree, as to shake the whole building by its shock; for which reason this practice should be avoided.

THEOREM.

486. *If two weights P, Q, are fixed to a crooked inflexible line ADF; they will still be reciprocally as the distances of their directions from the point of suspension, if they are in equilibrio. Fig. 14.*

Let the strait part *AD* be produced so as to meet the direction of the weight *Q* in *B*; then as it is the same thing whether the weight *Q* be fixed to the point *F* or *B*, it is evident that the weight *P* is to the weight *Q* as (*art.* 479) *BC* is to *CA*, if they are in equilibrio.

Of the LEAVER.

The Leaver is an inflexible rod or beam, supported in a point by a prop, which serves to raise weights: When the power is applied to one end, the weight fixed to the other and the prop between them, it is called a leaver of the first kind, but when the weight is between the power and the prop, it is then called a leaver of the second kind: Some authors distinguish five different sorts without reason; for as the power is always applied to one end, either the prop or the weight must be at the other; and therefore there are but two sorts in nature.

The leavers are supposed to be void of gravity, tho' there is no such thing in nature; because if their weight was to be considered, it would render the de-

monstrations more tedious, without any advantage in practice.

THEOREM.

487. *In both kind of leavers the power and the weight are reciprocally as the distances of their directions from the prop.* Fig. 15.

I. Let AB be a leaver of the first kind, and suppose the direction of the power P to be parallel to that of the weight; then will $P : W :: CA : CB$, by *art.* 480. But if the direction of the power be oblique to that of the weight; it is plain, that the power will then be to the weight as the distance CA between the direction of the weight and the point C, is to (*art.* 476) the distance between the direction of the power P and the point C.

Fig. 16. II. Let BC be a leaver of the second kind; then if we consider the prop C, as another power R, which with the power P supports the weight; we shall have (*art.* 479) $P : R :: AC : BA$; and by composition $P : P + R :: AC : BC$. Now because the powers P and R support together the weight W, the sum of their efforts must necessarily be equal to that weight; if therefore we substitute the weight W, instead of the sum $P + R$ of the powers, in the last proportion, we shall have $P : W :: AC : BC$.

If the direction of the power P be oblique to that of the weight, the power P will still be to the weight W, reciprocally as the distance between the direction of the weight and the point C, is to the distance between the direction of the power and the same point C; since the momentum of any power (*art.* 480) is in the compound ratio of its force, and the distance of its direction from the point fix.

488. Since in both leavers the power is always to the weight reciprocally as the distance between the direction of the weight and the prop, is to the distance between the direction of the power and the prop; a power applied to either leaver will have the same advantage, if the distance of its directions from the prop is the same.

N. B.

N. B. Upon this principle is founded the great use of handspikes, and other simple machines of the same nature; though the advantage seems to be the same, whether a power be applied either to the first or second kind; nevertheless, when the prop is between the power and weight, the power will be able to raise or support a greater weight than if the prop were at the end; because the weight of the lever conspires with the action of the power in that case, whereas it acts in a contrary direction in the other.

Fig. 17. There is often used a long lever AB, with a prop CE fixed to it by means of an iron pin C, about which it turns; when this lever is used in artillery, or upon any other occasion to raise great weights, it is from 12 to 15 feet long, 3 or 4 inches diameter; but when it is only for raising small weights, the lever is not above five feet long.

Plate XVII. Fig. 18. There is a machine used in Germany called Hebstock, made of one single piece of wood AB, with a slit or opening in the middle, and holes across placed diagonally, which serves to raise great weights in the following manner. A lever GH, with a wooden handle of about two feet long, and a hook with two or three links fixed to the end, is placed in the opening, and fastened on the opposite side to the burden with the hook; this done, there are two iron pins F, one of which is put into a hole under the lever, as high as possible; then the end is raised so as the other pin may be put into the next hole, above the former; after this the end is pressed downwards, and the first pin is put higher: Thus, by raising and depressing alternately, the end D of the lever, the burden may be raised to any height required.

This machine is very simple, and costs but little; the country people make them themselves, and is of excellent use in loading of large timbers on waggons, which a single man and a boy may do; it may serve upon all occasions where a hand jack is used; and as it is more simple and easier repaired, it is better and preferable to it.

Of

Of the P U L L E Y.

The Pulley is a small wheel made of wood, or brass, and turns about an iron pin called the axis; which, when the center is fixed to an immoveable object, is called immoveable, and moveable when the weight is fixed to the center.

T H E O R E M.

489. *If a power P sustains a weight W, by means of a rope passing over an immoveable pulley AB, in any direction, provided it be in the plane of the pulley; the power will sustain the whole weight. Fig. 19.*

For it is the same thing, whether the power and the weight are fixed to the extremities of the diameter AB, or the rope passes over the pulley; since the power will, in both cases, have the same advantage: Therefore this pulley may be considered as a lever of the first kind, where the prop is the point C or axis of the pulley. And since the radii, drawn perpendicular to the directions, express their distances from the point C, and are equal by the property of the circle; it is evident, that the power P is (*art.* 487) equal to or sustains the whole weight W.

490. Hence, if the power P raises the weight W by means of this pulley, the space through which the weight ascends, is equal to the space through which the power descends; since one end of the rope is as much lengthened as the other is shortened.

Hence it appears, that this pulley is of no advantage in respect to the increase of power; yet it is nevertheless very useful in many occasions: For it often happens that the direction of a power is not convenient, and by means of this pulley it may be changed into any other. Horses can draw only in a horizontal direction, which cannot always be had but by means of this pulley; and if a man was to carry a burden to a great height, it will be more convenient to do it by means of this pulley than to carry it upon his back.

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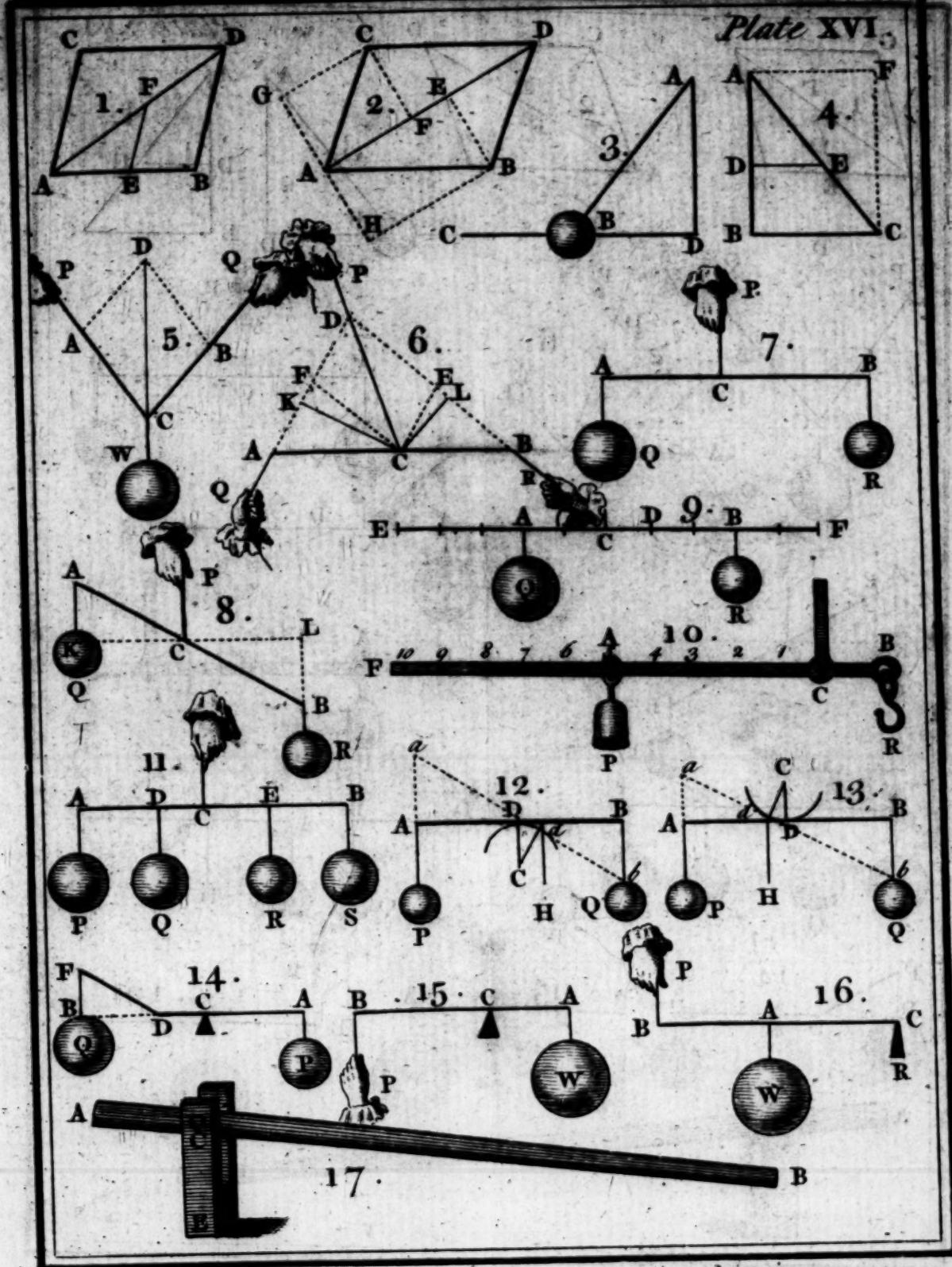
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It may be observed, that the diameter of a pulley may be great or small, without changing the equality between the power and weight; but the greater it is in respect to that of the axis, so much less will the friction be.

T H E O R E M.

491. *If a power P supports a weight W by means of a moveable pulley AB, in a parallel direction, it will sustain but half the weight. Fig. 20.*

If the rope, fixed to an immoveable object D, passes under the moveable pulley AB, and over the fixed one EF; then because it is the same thing, whether the point or extremity B of the diameter AB is fixed, and the power applied to the other A in a direction parallel to that of the weight, or that the rope passes under the pulley; it is evident that this pulley may be considered as a lever of the second kind, where the prop is at one end and the power at the other; and therefore (*art. 487*) $P : W :: CB : AB$, in case of an equilibrio; and because the radius CB is half the diameter AB by the nature of the circle, the power P will sustain but half the weight W by equality of ratios.

It may be observed, that by passing the rope over the first pulley EF, does not change the proportion between the power and the weight; since that pulley does not increase the power, it only changes the direction from upwards to downwards, which is more convenient, and often necessary.

492. It is evident that if the power P moves the weight W, by means of this pulley; the space through which the weight ascends is but half that through which the power descends: For if the weight W is raised one inch, the rope BD is shortened by an inch, as well as the part AF; and therefore the part EP is lengthened by two: And, in general, whatever space the weight W passes over, the parts BD and AF are shortened by as much, and the part EP is consequently lengthened by twice as much.

493. If

493. If the direction of the power instead of being parallel was oblique, as AG to that of the weight, by drawing BH perpendicular to AG; it is plain that the power P is to the weight W, as (*art.* 476) CB is to the perpendicular BH, to the direction of the power; since BH expresses the distance of the direction of the power, from the point B, which may here be considered as fixt.

T H E O R E M.

494. *If a power P sustains a weight W by means of a windless containing several pulleys, one half of them being fix and the other moveable, so as all the parts of the rope are parallel to each other; the power P will be to the weight W, as unity to the number of all the pulleys.* Fig. 21.

Because the moveable pulleys sustain the weight all together, they will support equal parts; so that if there are three, each will support one third of the weight; if there are four, one fourth; and so on in the same proportion; and since the power sustains but one half of the weight by means of one moveable pulley, it will sustain one fourth, by means of two; one sixth by means of three; and if there are four, each of them will sustain one fourth; and the power one half of this, that is, one eighth. Consequently, as the number of immoveable pulleys is equal to that of the moveable ones, the power is to the weight as unity to the number of all the pulleys.

495. Hence, if a power P raises a weight W, by means of a windless composed of any number of pulleys; the ascent of the weight will be to the descent of the power, as unity to the number of all the pulleys. For suppose the weight to raise one foot, it is plain that each end of the rope of the moveable pulleys must be shortened by a foot: and as there are as many pulleys in all, as there are ends, the power P must descend as many feet as there are pulleys; and through whatever space the weight ascends, each end must be shortened by as much, and the power P descends thro' a space, twice as much as there are moveable pulleys. Consequently, the space moved over by the weight is to the space moved

moved over by the power, as unity to the number of all the pulleys.

It has been found by experiments that a man may raise 150 pounds; if this force be applied to a windless of three moveable pulleys, he may raise 6×150 or 900 pounds. By this one may easily conceive to what degree a force may be increased by means of these pulleys; but it must be considered that what is gained in power is lost in time; which, however, is convenient on many occasions, when power is wanting and not time.

Of the W H E E L.

T H E O R E M.

496. *If a power P be applied to a wheel in a direction of a tangent to its circumference, and sustains a weight W, fixed to its axis; the power will be to the weight reciprocally as the radius CA of the axis is to the radius CB of the wheel. Fig. 22.*

It is evident, that whether the power be applied to the extremity B of the line AB, and the weight to the other A, or to the ropes which pass over the wheel and round the axis, the power will have the same advantage; and as the line AB may be considered as a lever of the first kind, where the prop C is between the power and the weight; the power P will be to the weight W, as the radius CA of the axis is to the radius CB of the wheel.

497. Hence. if the power P raises the weight W by means of the wheel, the ascent of the weight will be to the descent of the power, as the radius CA of the axis is to the radius CB of the wheel: For since the wheel and axis move together, the space moved over by the weight in one turn of the wheel, will be to the space moved over by the power in the same time, as the circumference of the axis is to the circumference of the wheel; but because the circumferences of circles are as their radii, the space moved over by the weight

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is to that moved over by the power; as the radius CA of the axis is to the radius CB of the wheel.

498. If the direction of the power P is not in a tangent to the circumference, but in any other as DE; then by drawing CE perpendicular to DE, we shall have (art. 476) $P : W :: CA : CE$. Therefore the power P acts to the greatest advantage when its direction is a tangent to the circumference of the wheel.

The wheel is very useful on many occasions; such as in drawing water out of wells, either by hand or with horses; stones out of pits; earth out of mines: In some cases the circumference is omitted, and the power is applied to the extremities of the radii; the machine is then called a Cabstan.

Of the inclined PLANE and WEDGE.

T H E O R E M.

499. If a power P sustains a weight W by means of an inclined plane AB, in a direction DP, parallel to the plane; the power will be to the weight as the height BC of the plane is to its length AB. Fig. 23.

Through the center of gravity D of the weight, draw the line DE perpendicular to the base AC, and DF, to the plane; from any point in DE draw EF parallel to AB, and FG to ED; then if the side DE expresses the weight W, the diagonal DF of the parallelogram EG will express the part sustained by the plane, and EF that which is supported by the power P; for whilst gravity acts in the direction of the side DE, the power in DG, the plane resists in the direction FD of the diagonal. Now because DE is parallel to BC, and EF to AB, the angle E will be equal to the angle B; and since DF is perpendicular to AB, it will also be perpendicular to its parallel EF: Therefore the triangle DEF is right angled at F and similar to (art. 69) the triangle ABC; and so $EF : ED :: BC : AB$; but the weight W has been proved to be to the part sustained by the power P,

P, as DE is to EF. Consequently $P : W :: BC : AB$, by equality of ratios.

500. Hence, if a power P raises a weight W, by means of an inclined plane, the vertical ascent of the weight is to the space described by the power as the height of the plane BC is to its length. For it is evident, that whilst the power moves the weight from A to B, it describes the same space AB, and the weight is raised from C to B.

T H E O R E M.

501. If a power P sustains a weight W by means of an inclined plane ABC, in a direction GP parallel to the base AC; the power will be to the weight as the height BC of the plane is to its base AC. Fig. 24.

By completing the parallelogram EG in the same manner as before, only so as EF be parallel to the base AC; then if DE expresses the weight, EF will express the part sustained by the power P, and DF that which is sustained by the plane: And because DE is parallel to BC, and EF to AC, the angle E will be a right one; and since DF is perpendicular to AB, it will make the angle EDF, with DE, parallel to BC, equal to the angle A; hence the triangles ABC and DEF will be similar: Therefore (art. 69) $EF : ED :: BC : AC$; but we have proved that the power is to the weight as EF is to ED. Consequently $P : W :: BC : AC$.

502. Hence, if the power P raises the weight W, by means of an inclined plane, in a direction parallel to the base AC; the space moved over by the power is to the vertical ascent of the weight as the length of the base AC is to the height BC of the plane; for whilst the power P moves the weight W from A to B, it moves itself through the space AC, and the weight ascends from C to B.

N. B. If we suppose an equal plane was joined to this, so as to have the same base AC, we shall have the wedge; and the force in the direction CA, to drive it into a piece of wood, will be to its resistance as twice
EF.

EF is to twice DF, or as the height BC is to the length AB.

Vouffoirs of arches may be considered as so many wedges which endeavour to descend by their gravity; and therefore the proportion between the resistance and the weight of any one, may be determined when its position is given.

Of the S C R E W.

Although the screw is well known, yet it will be necessary for the better understanding its application, to shew in what manner the thread of it may be formed.

Fig. 25. Suppose the height AB of a cylinder ABCD to be divided into any number of equal parts, Ba , ab , bc , cd , and lines ae , bf , cg , db , drawn perpendicular to the side AB, each equal to the circumference of the base; by joining Be , af , bg , cb , there will be as many right angled triangles Bae , abf , bcg , cdb , as the height of the cylinder has been divided into equal parts.

Now if we imagine these triangles to be rolled upon the cylinder, it is evident that the point e will coincide with the point a ; f with b ; g with c ; b with d ; and therefore the hypotenuses Be , af , bg , cb , will form on the surface of the cylinder one continued spiral line, which represents the thread of the screw.

T H E O R E M.

503. *If a power applied to a screw sustains a weight in a direction perpendicular to the axis and quite close to the circumference; the power will be to the weight, as the interval between any two contiguous threads is to the circumference of the screw.*

Let the power P be applied to the point B in the direction BC ; and suppose the screw to be unfolded so as to form the inclined planes Be , af , bg , cb ; then it is plain that the power is to the weight as the (*art. 501*) height aB of the plane is to its base ae ; but the base af is equal to the circumference of the screw by the generation

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tion of it, and the height aB of the plane equal to the distance between any two contiguous threads. Therefore the power is to the weight as the interval between any two contiguous threads is to the circumference of the screw.

504. If a power raises a weight by means of a screw, the space through which the weight ascends is to the space described by the power as the interval aB between any two contiguous threads is to the circumference ae of the screw.

Fig. 26. As the power is never applied close to the screw when it is made use of to raise weights, a leaver GH is passed through a hole made in the nut GI , and by turning it round it presses against a strong board, by which the screw presses downwards or upwards, according as the nut is below or above the board: It is plain, that in this case the power is to the weight as the interval between any two contiguous threads is to the circumference described by the point H , of the leaver, where the power is applied.

If the leaver GH be four feet, the circumference described by the point H will be 25 feet nearly, or 300 inches; and if the interval between two contiguous threads be an inch, the power will be to the weight as unity is to 300: Now if a man can carry 130 pounds, he may, with the same force applied to a screw, raise 130×300 , or 39,000 pounds nearly; for this is the weight which would be in equilibrio with that force, setting aside all friction and obstacles: Which shews to what prodigious degrees a power may be increased when applied to a screw.

THEOREM.

505. If a power P sustains a weight W , by means of an endless screw AB , with a handle ACD to it; and this screw turns a toothed wheel E , and the pinion L of it, another wheel, and this continued to any number; and the weight W be fastened to the barrel N of the last wheel; the power will be to the weight as the product of the dis-

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distance between any two contiguous threads, multiplied by the radii of all the pinions and the barrel, is to the product of the circumference described by the handle CD, multiplied by the radii of all the wheels. Plate XVIII. Fig. 27.

If a power N were applied to the circumference of the lower wheel G, in the direction of a tangent; this power N would be to the weight W, as the radius of the barrel N is to the radius of the wheel; and if a power M were applied to the circumference of the second wheel F, it is evident that the power N may be considered as to be applied to the pinion M; and so the power M will be to the power N, as the radius of the pinion M is to the radius of the wheel F: In the same manner, if a power L were applied to the third wheel E, that power will be to the power M as the radius of the pinion L is to the radius of the wheel E.

Lastly, the power L may be considered as applied to the endless screw; and therefore the power P will be to the power L as the interval between any two contiguous threads, is to the circumference described by the handle.

Now if the radii of the barrel N, and the pinions M, L, be n, m, l , respectively, those of the wheels G, F, E; g, f, e ; the distance between any two contiguous threads of the screw d ; and the circumference described by the handle c ; we shall have

$$N : W :: n : g.$$

$$M : N :: m : f.$$

$$L : M :: l : e.$$

$$P : L :: d : c.$$

$$P : W :: n m l d : g f e c.$$

506. If the power P raises the weight W, by means of these wheels and the screw, the space described by the power P will be to the space described by the weight W, as the circumference described by the handle multiplied by the number of revolutions it makes, whilst the barrel N turns once to the circumference of the barrel; or because

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cause the circumferences are as their radii, as the radius of the handle multiplied by the number of revolutions to the radius of the barrel.

If the number of teeth in the wheels and the pinions are proportional to their radii or circumferences; then the space moved over by the power will be to the space moved over by the weight W , as $gfec$ is to $nml d$.

Examp. If the radii of the wheels be 12 inches each, those of the pinions and barrel one, the interval between two contiguous threads one; the radius AC to be 1 foot or 12 inches, its circumference will be 75 inches nearly; therefore $P : W :: 1 : 129600$. in case of equilibrio.

By this last theorem we may compute the increase of power when applied to any wheel-work, as a hand-jack, all sorts of engines, and also to the block carriage used in artillery.

T H E O R E M.

507. *If a Power P be applied to the extremity D of a lever, which turns an horizontal capstan, and the weight W be fastened to the axis AF , by means of a rope; the power will be to the weight as the radius of the axis is to the length FD of the lever, in case of equilibrio. Fig. 28.*

We suppose the power to be applied in a direction perpendicular to the lever; and therefore, this machine may be considered as a wheel in its axis, the lever FD representing the radius.

To this machine may be referred the Gin, used to raise heavy burthens in artillery, such as guns and mortars; to the drawing stones out of deep pits; or earth out of the shafts of mines; as likewise to the sling-carts: The only inconveniency there is in this machine is, when the lever is nearly vertical; then the power cannot be applied in a direction perpendicular to it; for which reason the vertical one, such as is represented by fig. 29, is generally preferred when it can be done.

Fig. 29. Cranes are made in this manner, as likewise the boring engine, and many others of that sort; the

leavers may here be made of any length, without being the least burthen to the power.

As the science of mechanics is the most extensive, and at the same time the most useful part of the mathematics, it would require several volumes to explain its various uses; for which reason we shall refer the reader for farther enquiries of this kind to Mr. *Belidor's* works, where he will find the greatest variety of applications to machines that can be met with in any author whatsoever.

WHEEL CARRIAGES.

The common practice of placing the spokes of a wheel obliquely on the nave, which the workmen call *Dishing*, seems defective; since if they were perpendicular, they would in appearance be stronger and better able to support the weight on the carriage, than being oblique: As no author has treated of it, nor the workmen can give any reasons for it: I shall give the following ones.

Fig. 30. If the spokes AC, instead of being oblique were perpendicular, as CD to the nave KL, they would certainly be in the best position, provided the carriage did always move on an horizontal plane, but when it moves on an inclined one the direction would then be oblique to that of the weight which they support, and so would lose their advantage of being perpendicular. And since when the carriage moves on an horizontal plane, each wheel supports but half the weight, whereas on an inclined one the lowest supports so much the more as the plane has a greater inclination: It follows, that if the spokes were perpendicular to the nave, they would be oblique when one wheel is higher than the other; and therefore have the least strength where it supports the greatest weight. Consequently, the spokes are more advantageously placed when oblique, than if they were upright.

Because the planes on which carriages move are variously inclined, it was necessary to fix upon some inclination or other, as a mean between them, the work-

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men make them incline 3 inches in a wheel of 5 feet high; that is, if from the point C of the nave, CD be drawn perpendicular to the horizontal line AF, the part AD is three inches; from whence the angle ACD of inclination is found to be 3 degrees and 50 minutes: If AE is perpendicular to the spokes, it is evident that the angle EAD, which expresses the inclination of the plane when the spokes become perpendicular, is equal to the angle ACD.

Being willing to know the greatest inclination of a plane, where a carriage may just move upon without oversetting, I found this angle, by supposing the center of gravity 10 inches above the center of the axletree, to be about 8 degrees and 45 minutes; which shews, that the angle which the spokes make with the nave is nearly half the angle of the greatest inclination.

Fig. 31. The next thing to be considered in wheels is their height, which varies pretty much; those who are for high wheels say, that if they meet with an obstacle such as the inclined plane LAB, they do not require so great a force to draw them over as low ones do; but whether wheels are moved in a plane or up hill, the direction of the draught will be the same; so there is no reason to prefer one height before another; it is true, that when a stone, or something of that kind, happens to be in the way, the high wheels require less strength to draw them over than the low ones; but as this happens but seldom, and by accident only, it is not of sufficient consequence to regulate the heights of wheels thereby.

It has been disputed, whether a high or a low draught is best, which in my opinion is easily decided; for it is manifest, that if the line CH of direction, in which the horse draws, is parallel to the horizon, it will be the most advantageous of all; since if the wheels are higher, a part of the force is lost by the pressure on the ground, and when they are lower a part is lost in lifting the weight upwards.

Fig. 32. It remains now to determine the figure of

the axletree; some will have the axis or center line of the arms to be in one continued right line; and others, that the lower parts AB, CD, should be so: It is plain that when the lower parts AB, CD, are in a continued right line, the wheels will roll on an horizontal plane; and therefore neither endeavour to fly off and cause a great friction against the linch pins, nor to approach the body of the axletree, and cause a friction on that side; whereas if they bend upwards at the ends A, D, the wheels will roll on an inclined plane, by which they cause a friction against the linch pins, and if they either break or fall out, the wheels fly off in an instant, and the carriage drops down: The contrary happens when the ends bend downwards; for then the wheels rub against the body of the axletree, and cause thereby a great friction: It is therefore necessary, that in all heavy carriages, such as are used in artillery, the lower part of the arms should be in one continued right line. As to the objection made by some, that the wheels are wider above than below, whereby they become oblique, which is not so well as if the arms are so made as that the wheels are upright; I say this objection is inconsiderable, in regard to the advantages which such carriages have above others. In light carriages, where there is no great difference between the bigness of the arms, it is not necessary they should be made so, the pressures against the linch pins being then but small.

S E C T. III.

P R O J E C T I L E S.

THE projectiles we are going to treat of are supposed to be described in a non-resisting medium, tho' we know by experience that the air resists considerably; for at *la Ferre* we made many hundreds of experiments, and always found that the ranges, when short, agreed very nearly with those found by the theory, but always shorter in long ranges; for which reason the skillful bombardiers makes a competent allowance

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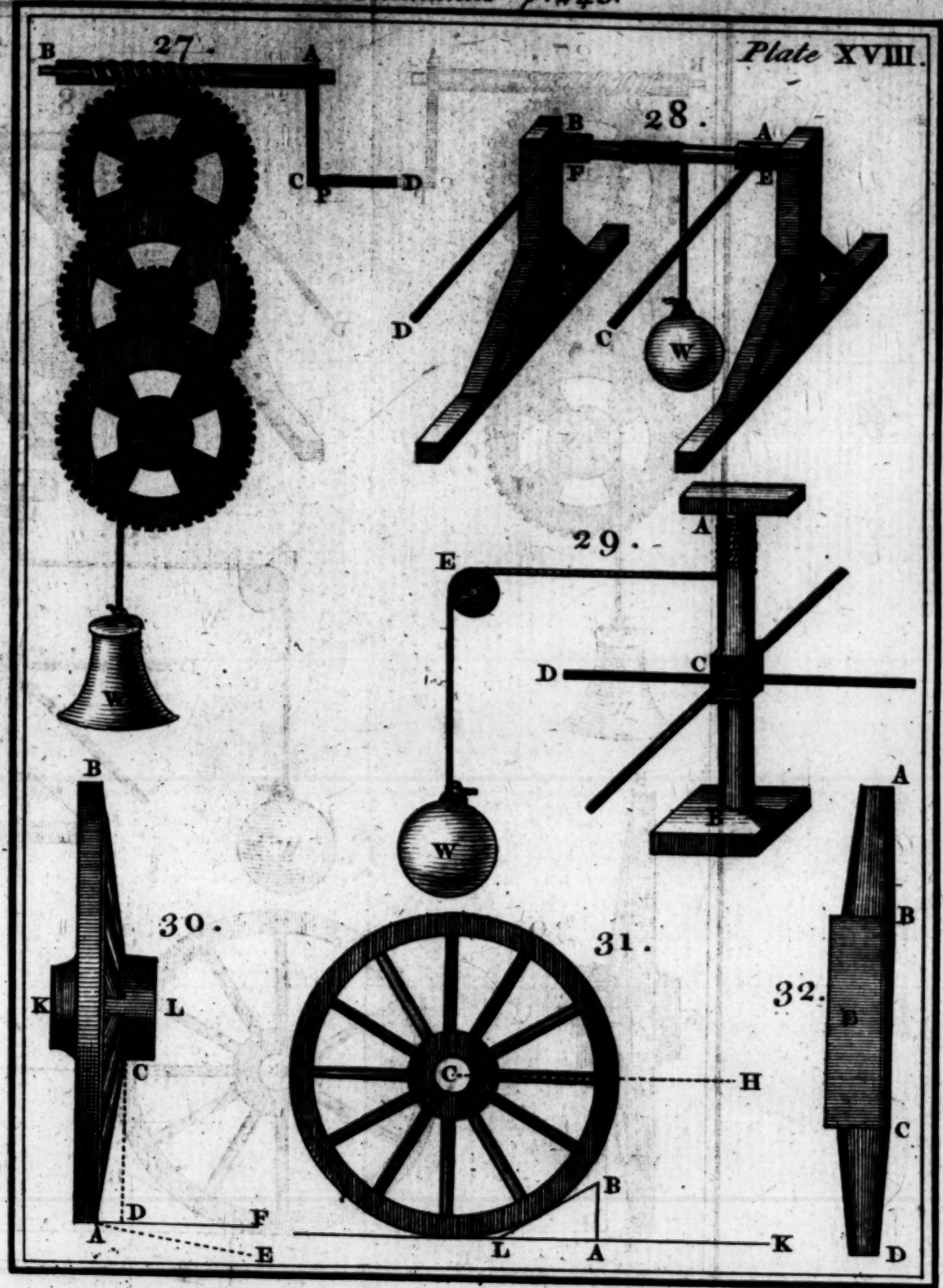
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PLATE XVIII.

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Many are the authors who have treated on this subject, who will tell you in their prefaces, that they rendered the subject easy, so as to be understood by those practitioners who have no knowledge in the mathematics; but, for want of having the practice themselves, fall generally short of their promise.

P R O B L E M.

508. *If a body be projected from the point A in the direction AT with a given velocity, it is required to find the path ABCD described by this body.* Plate XIX. Fig 1.

If the given velocity be such, that the body may describe uniformly the spaces AG, AH, in the same time, it descends thro' the vertical spaces GB, HC, by the force of gravity, it will describe by these joint forces (*art.* 440) the arc AB of a curveline: And because spaces described uniformly with the same velocity, are (*art.* 433) as the times of their description, AG is to AH, as the time in which AG is described is to the time in which AH is described; or because AG, GB, and AH, HC, are described in the same time by supposition, AG is to AH as the time in which the body descends thro' the space GB by the force of gravity, to the time in which it descends through HC by the same force: But the spaces described by falling bodies, are as the (*art.* 450) squares of the times during their fall: Therefore the square of AG is to the square of AH, as GB is to HC. Consequently, the curve ABCD is such, that by drawing lines GB, HC, TD, we have always $AG^2 : AH^2 :: GB : HC$. Which is the property (*art.* 270) of the parabola.

509. Because the space described uniformly, with the velocity acquired by falling through GB, in the same time is (*art.* 451) double the space GB; and the spaces described uniformly in the same time (*art.* 434) as the velocities with which they are described; it follows that AG is to 2 GB, as the projectile uniform velocity at A is to the accelerated velocity at B.

510. If AK be the height thro' which a falling body may acquire the projectile velocity; then because the

spaces described by falling bodies, from a state of rest, are as (*art.* 450) the squares of the velocities acquired during the fall; AK will be to GB as the square of the projectile velocity at A is to the square of the accelerated velocity at B: But the square of the velocity at A is also to the square of the velocity at B, (*art.* 509) as the square of AG is to the square of 2GB. Therefore $AK:GB::AG^2:4\times GB^2$, by equality of ratios; or $AK:AG::AG:4GB$ by division and multiplication. And by the same reason $AK:AT::AT:4ID$.

P R O B L E M.

511. *The velocity with which a body is projected in a given direction At, being given, to determine the horizontal range AB. Fig. 2.*

Erect AK, perpendicular to AB, and equal to the height thro' which a falling body may acquire the given velocity; with AK, as a diameter, describe the semicircumference KNA, and thro' its intersection e, with the given direction, draw eD perpendicular to AB; then if AB is made equal to 4AD, the line AB will be the range required.

For if Ke be drawn, then because AD is perpendicular to AK, the line AD touches (*art.* 111) the circle in A; and hence the angles AKe, DAe, which are both measured by half the same arc Ae, are equal; therefore the right angled triangles AeK and ADe are similar, and $AK:Ae::Ae:eD$; but if Bt be drawn perpendicular to AB, and meeting the direction At in t, the similar triangles ADe, ABt, give $Ae:eD::At:tB$; whence we get $AK:Ae::At:tB$, by equality of ratios: Now if the consequents are multiplied by 4, and we take for 4Ae its equal At, we shall have $AK:At::At:4tB$. Consequently the curve passes thro' the (*art.* 510) point B.

512. Hence, if the horizontal range AB be given, and the projectile velocity, the line of direction At, in which the body being projected with the given velocity, shall hit the given point B, may be found: For if AD be the fourth part of AB, and DE drawn per-

perpendicular to AB, and thro' its intersection *e* with the circumference, a line At, then this line will be the direction required, since it is proved in the same manner as in the last problem that $AK : At :: At : 4tB$.

513. If the body be projected with the same velocity in different directions, the horizontal ranges AB will be as the sines of angles double those of elevations: For if the radius ON be drawn perpendicular to the diameter AK, intersecting De produced in L; then will OL, or its equal AD, be the sine of the arc Ae, which measures an angle double the angle eAD of elevation; and since AD is always the fourth part of the horizontal range AB, it is evident when AK remains the same, the sines AD of angles double those of elevations, will always be in the same proportion to one another as the horizontal ranges.

514. If the line De intersects the circumference in two points *e*, E, it is manifest that there are two directions At, AT, in which the body being projected with the same velocity, acquired by falling thro' KA, will hit the same point B in the horizontal line AB: For the triangles AEK, ADE, are similar; and therefore $AK : AE :: AE : ED$, and $AK : AE :: 4AD : 4ED$; or $AK : 4AE :: 4AE : 16ED$: But if the line AT was produced, so as to meet Bt produced, it would be equal to 4AE; because AB is 4AD, and Bt produced 4DE. Consequently the body would hit the point B.

515. Hence, because DE is parallel to AK, the arcs Ae, KE, are equal; it is evident, that if the direction in which the body is projected, makes the same angle either with the vertical line AK, or with the horizontal line AB, that body will have the same horizontal range.

516. If the line DE instead of cutting the circle only touches it in N, the angle NAB of elevation being then measured by half the quadrant AeN, will be 45 degrees; and the horizontal range, which is 4AD, will become four times the radius, or twice the height AK, and is the greatest that can be attained with the same projectile force.

517. If

517. If the arc Ae , or KE be 30 degrees, its sine OL , or AD will be equal to half the radius AO ; and the horizontal range AB , which is quadruple of AD , will be equal to the diameter AK . Consequently, when the angle of elevation is 15 degrees, the horizontal range is then just equal to the height thro' which a falling body acquires the projectile velocity; or half the greatest range (*art.* 516) that can be attained by the body with the same projectile force.

P R O B L E M.

518. *If a body be projected in a given direction At, with a given velocity, to find the highest point to which that body will ascend.*

Suppose the force of projection in the direction Ae to be reduced into the horizontal one AD , and into the vertical one De ; so that by the first AD , it is carried from A to B ; and by the second De it rises upwards; but as the force of gravity constantly acts in a contrary direction to the force De , it will in equal times destroy equal parts of that force, and the body will rise upwards till the whole force in that direction is destroyed; then it will descend by the force of gravity till it arrives at B , where it will have acquired a force equal to that which was destroyed; therefore the ascent and descent thro' the same vertical space being made in equal times, the body will arrive to the highest point in half the time that the projectile is described; but because a body would fall through the space Bt , in the same (*art.* 513) time that the projectile is described, it will fall thro' one fourth of that space in half the time; since the spaces described by falling bodies from a state of rest are as the (*art.* 450) squares of the times, and as AD is one fourth of (*art.* 516.) AB , De will also be one fourth of Bt . Consequently De is the greatest height to which the body ascends in describing the projectile.

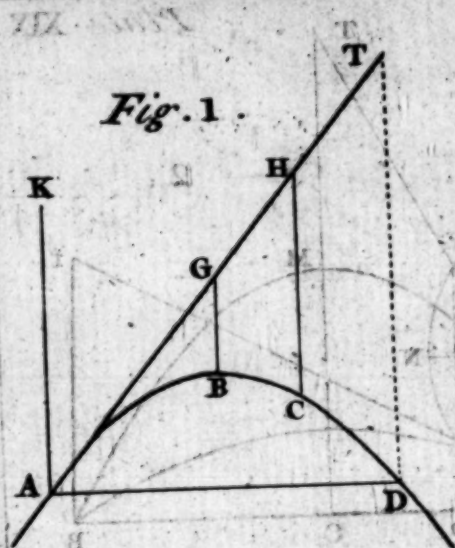
519. It is evident that De is the versed sine of the arc Ae ; and because the altitude of the projectile is
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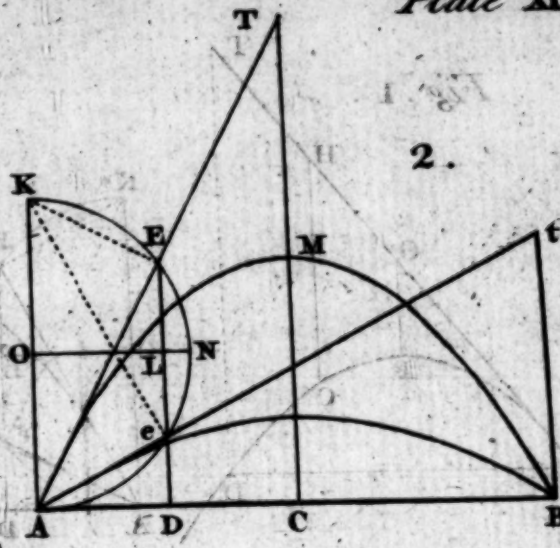
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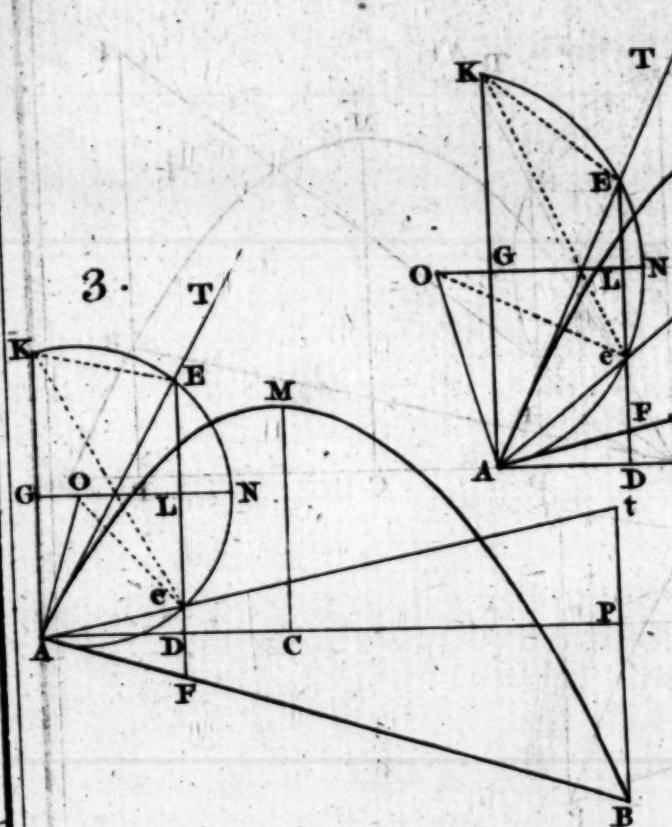
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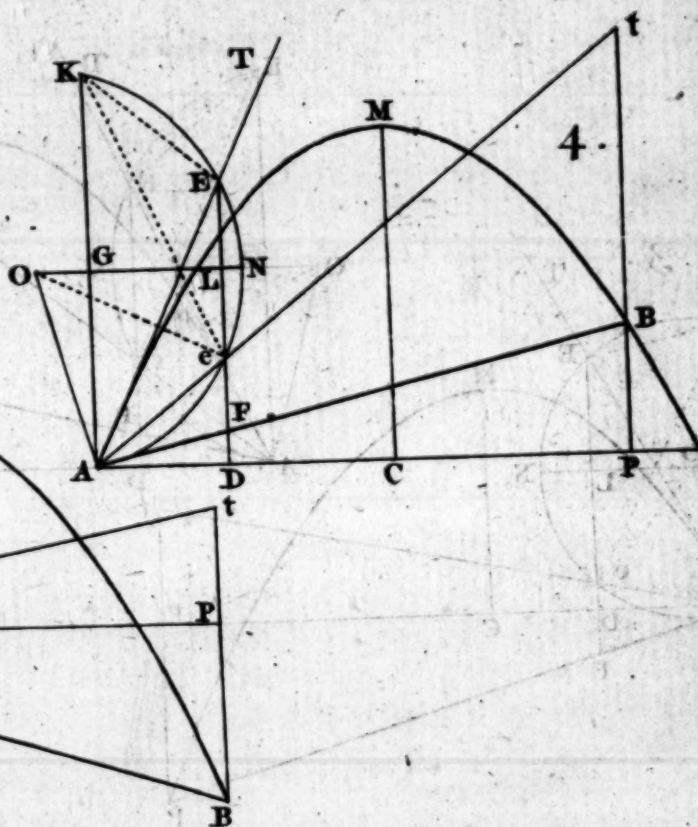
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equal to that line, it follows that the altitudes of projectiles described with the same force, are to one another as the versed sines of arcs double those which measure the angles of elevation.

520. Since spaces described by falling bodies from a state of rest are as the squares of the times (*art. 450*) in which they are described; the time in which a body would fall thro' AK , is to the time in which a body would fall thro' eD , as AK is to Ae : For since the lines AK , Ae , and eD , are (*art. 516*) in a continual proportion, AK is to De as the square of AK is to (*art. 76*) the square of Ae ; and as the square of the time of the body falling thro' AK is to the square of the time of the body falling thro' eD , as AK is to De , or as the square of AK to the square of Ae by equality of ratios. Consequently, the time in which a body would fall thro' AK , is to the time in which a body would fall thro' eD , as AK is to Ae .

521. Because the time in which the body describes the projectile is double the time (*art. 523*) in which the body would fall thro' eD ; it is evident, that the time a body would fall thro' AK , is to half the time in which the projectile is described as AK is to Ae . Therefore the times in which projectiles are described with the same force, are to each other as the chords Ae of angles double those of elevations; or because these chords are as the sines of half the angles which measure the arcs; the times will be as the sines of the angles of elevations.

522. It is also manifest, that if a body be projected with an angle of elevation of 45 degrees, with different forces, the horizontal ranges will be to each other as the squares of the times in which they are described: For when De only touches the circle in N , De and AD become each equal to the radius AO ; and the horizontal range AB is then equal to four times the radius; and since the line eD is described in half the time that the projectile is described, a body would fall through a height double the height AK in the (*art. 450*) same time that the projectile is described; that is, it would fall thro'

thro' a height equal to the horizontal range AK. Consequently, the horizontal ranges described with an angle of 45 degrees are as the squares of the times.

P R O B L E M.

523. *The inclination of a plane AB either above or below the horizon AP, the line of direction At, and the projectile force being given; to find the point B where the body will hit that plane. Fig. 3, 4.*

On the horizontal line AP erect AK perpendicular, and equal to the height through which a falling body may acquire the given velocity; bisect the line AK by the perpendicular ON, and draw AO perpendicular to the inclined plane AB; then if from the center O, an arc of a circle be described through the points A, K, and through its intersection *e*, with the given direction, *eD* perpendicular to AP; by taking AP quadruple to AD, the perpendicular PB to AP will determine the point B required.

For because AB is perpendicular to the radius AO, it will be a tangent to the circle; and so the angle BAE is equal to the angle AKe, as well as the alternate ones KAe, AeD; so that the triangles AKe, AeF, are similar, and therefore $AK : Ae :: Ae : eF$; but the parallels Fe, Bt, give $Ae : eF :: At : tB$; whence, by equality of ratios we have $AK : Ae :: At : tB$; and since AP is the quadruple of AD, AB will be the quadruple of AF, and At quadruple of Ae; if therefore we take the quadruples of the consequents we shall have $AK : 4Ae$ or $At :: At : 4Bt$. Consequently, the projectile (*art. 515*) passes thro' the point B.

524. Hence, if the range AB, the inclination of the plane and the projectile velocity, are given; the line of direction At will be found by taking AD equal to one fourth of AP, or AF to one fourth of AB; by drawing De parallel to AK, and At through its intersection *e* with the arc ANK.

525. When the line De intersects the arc in two points, there will be two directions At, AT, in which the body being projected with the same velocity, shall
hit

hit the same point B in the inclined plane; and when that line only touches the arc in N, the range will then be the greatest possible that can be attained with the same velocity.

526. Because AK is perpendicular to AP, and AO to AB, the angle OAK is equal to the angle of inclination PAB of the plane; and the angle NOA its complement. Consequently, the angle of elevation, which corresponds to the greatest range, is half the complement to the angle of inclination of the plane, when the plane is above the horizon; or to half the supplement (fig. 3) of the complement, when the plane is below the horizon.

P R O B L E M.

527. *The point B on a plane AB whose inclination is given, together with the projectile velocity being given; to find the angle of elevation DAe such as the body shall hit the given point B.*

Because the angle BAP, or its equal OAG, is given as well as AG, being half the height AK, thro' which a falling body acquires the given velocity; therefore as the radius is to the tangent of the angle OAG, so is AG to GO; and as the radius is to the secant of that angle, so is AG to AO or Oe; again, as the radius is to the cosine of the same angle, so is AF the fourth part of the given distance AB to AD, or its equal GL, to which add OG in fig. 4, or subtract it in fig. 3; then the sum or difference OL will be the cosine of the angle NOe, which is the difference between the given angle AON and the angle AOe, or FAe, to which the angle DAe being added in fig. 3, or subtracted in fig. 4, gives the angle DAe of elevation; which consequently is known.

PRACTICAL RULES for HORIZONTAL RANGES.

I. *The range of a body projected with an angle of 15 degrees is half the range of that body, if projected (art. 522) with the same force under an angle of 45 degrees.*

II. *The*

II. *The range of a body projected with an angle of 45 degrees, is equal to the square of the time of its description expressed in seconds, multiplied by 16.1 feet.*

This rule results from the law of falling bodies, which describe spaces proportional to the squares of the times; and a body falling thro' the space of 16.1 in a second, and that the time of description of the projectile is (art. 527) equal to the time of a falling body describing a space equal to the horizontal range.

III. *If a body be projected with the same force and different angles of elevations; the horizontal ranges are as the sines (art. 518) of angles double those of elevations respectively.*

Examp. I. *Let a body projected under an angle of 45 degrees be 12 seconds in its flight: What is the horizontal range?*

The square of 12 is 144, which multiplied by 16.1 feet gives 2318.4 feet, or 772.8 yards, by rule the second, for the range required.

Examp. II. *If the range of a body projected with an angle of 25 degrees be 200 yards, what will be the range if the body be projected with the same force, with an angle of 30 degrees?*

The sine of 50 degrees, double of 25, is 76604, and the sine of 60, double that of 30, is 86602; therefore $76604 : 86602 :: 200 : 226$ = to the range required by the third rule.

Examp. III. *If the range of a body projected with an angle of 20 degrees be 200 yards; what must the angle of elevation be to project the body with the same force at a distance of 300 yards?*

The sine of 40 degrees, double of 20, is 64278; whence $200 : 300 :: 64278 : 96417$ = to the sine of an angle double the angle sought; this sine answers to an angle of 74 degrees and 37 minutes; half of which 37 degrees and 18.5 minutes, will be the angle required.

The second rule is useful at sea, where no trial of the force of the powder can be made; it will be sufficient to know the distance from the ship to the object, then the

time

time required for the shell to hit the object, must be found by saying, 16.1 feet is to the given distance as the square of one second is to the square of the time required, and extracting the square root, which gives the time, so that when the time of the flight of the shell is observed, you will see whether it exceeds or falls short of the time found, and the range will exceed or fall short of the given range. Those who are unacquainted with the practice, may imagine that one might see whether the shell falls in the place aimed at, and therefore the rule is useless; but in practice it is otherwise; for a ship may be at least a mile off from the object, and the bystander fancy that the shell did fall very near the place, when it may be 4 or 500 yards from it.

If it is proposed to throw a shell at a given distance, it will be most convenient to throw a shell with an angle of 15 degrees of elevation, then if twice the range is either equal to or greater than the given distance; that quantity of powder will be sufficient to throw the shell to the given distance, if not, another trial must be made with more powder; whereas if the first shell is thrown with any other angle of elevation, it is not so easily known, whether that quantity of powder is sufficient to throw the shell to the distance proposed.

It would be very useful to know the greatest ranges of all the different mortars, loaded with different charges; for having a table of these ranges, one might easily know what quantity of powder is sufficient to throw the shell upon an occasion to a given distance.

N. B. These are all the useful practical rules in gunnery, which are but few and plain; from whence it might be imagined that a shell may be thrown so as to hit any proposed object, without any difficulties; but it must be considered, that there are many unavoidable accidents in practice, which renders it very difficult; such as the inequality of the powder, unequally placed in the chamber, the mortar or bed not being truly made; the platform giving way in one place or other; the inequality of the weight and roundness of the shell; to
this

this we may add the resistance of the air, which is very great, and varies the ranges in proportion as they are long. However, the practitioner must not conclude from thence, that these rules are of no service, since there are so many obstacles which prevent their exactness; for tho' it is impossible to attain to a great exactness, yet notwithstanding, by taking all the care that a skillful bombardier is capable of, they will answer sufficiently in most cases, as I have found by experience.

We have not given any practical rules for throwing shells on inclined planes, because the theory being not very easy, besides the resistance of the air, alters the ranges so considerably, that it would be needless to enter any farther into this subject, referring the reader to our Treatise of Fluxions for the true figure described by the body.

SECT. IV.

HYDROSTATICS.

DEFINITIONS.

1. **H**YDROSTATICS is the science which considers the equilibrium and pressure of fluids.
2. A fluid is a body whose parts yield to any force impressed, and are easily moved one amongst another.
3. Specific gravity is a comparative term, and signifies the proportion between the gravities or weights of different bodies which have the same magnitude.

It must be observed that the weights or gravities of bodies are always equal to the quantities of matter contained in them; that is, the weight of a cubic inch of water is to the weight of a cubic inch of gold, as the quantity of matter contained in the cubic inch of water, is to the quantity of matter contained in the cubic inch of gold.

It seems also probable, that all the primary parts of matter are the same in all bodies; since solids become fluid by heat, and fluids solid by cold.

As

As there are hard bodies which are elastic, and others which are not; so there are fluids which are elastic, and others not. A fluid is elastic when it may be reduced into a less bulk by compression, such as air; and non-elastic, when it cannot be reduced by any force whatever, such as water; it is only of this latter sort we shall treat here in this section, and the former shall be considered in the next but one.

THEOREM.

528. *The surface of a fluid contained in a vessel will be always smooth and level, when it is acted upon by the force of gravity only.*

For since every particle is pressed downwards by the force of its gravity; should any one be higher than the rest it would endeavour to descend lower, and having nothing to support it, or to oppose its motion, will descend, till such time it becomes even with the rest, and remain so, unless some other force disturbs it.

THEOREM.

529. *If a siphon, or a tube with two branches AFB, be filled with an homogeneous fluid; the surfaces of this fluid in both branches will be in the same horizontal line.*

Plate XX. Fig. 1.

For if the fluid was put in motion by some external force, the quantity $AacC$ which passes through a section AC in one branch, must be equal to the quantity $BbdD$, which passes thro' a section BD in the other; and therefore the descent Aa in one, will be to the ascent Bb in the other, reciprocally as these sections AC , BD : But we have proved (*art.* 481) that if two bodies which are in equilibrio, are put in motion, the ascent of the one, and the descent of the other, are reciprocally as their weights.

And since the descent Aa in the branch ACR is to the ascent Bb , in the branch BDS , reciprocally as the section BD is to the section AC ; it follows, that the fluid in one branch is in equilibrio with the fluid in the other.

other. Consequently, the surfaces in both branches are in the same horizontal line.

Fig. 2. If one or both branches are oblique as BS, the demonstration will be the same, provided the perpendicular distance between the surfaces BD, bd , in the oblique branch, is taken instead of the length Bb.

§ 30. Hence, if the fluid be put in motion, the times of the ascents and descents in either branch are equal. For let Aa, Ae, be any two descents answering to the ascents Bb, Bb; then because the velocities with which the surface AC descends, and that with which the surface BD ascends, are reciprocally as these surfaces, as well as the quantities of the fluid, it follows, that the generating forces being as the velocities, the times in which these velocities are generated must be equal.

The same thing is true in regard to the waves of stagnated water.

T H E O R E M.

531. *An upright vessel filled with an homogeneous fluid is pressed every where by a force proportional to the surface pressed and the height of the fluid above it.* Fig. 3.

For let the height of the vessel ABCD be divided into any number of equal parts as Ca, ab, bD; then it is plain, that the surface na is pressed by the weight of the fluid nBCa, the surface mb pressed by the weight mBCb, and the bottom AD by the weight of the whole fluid.

Now the pressure of the fluid nBCa on the surface na is resisted by the fluid below it, because reaction is equal to action; it will therefore endeavour to move in that plane by a force equal to its weight, if it was not resisted by the sides of the vessel; and if there was a hole either at n or a, it would move through it with that force; and therefore the parts na, which prevent this motion, will sustain the whole pressure of the weight of the fluid nBCa above it; in the same manner the parts mb, of the plane mb will be pressed by the whole weight of the fluid mBCb above it. And since the weights of the

fluids

fluids $nBCa$, $mBCb$, which have the same base, are as the altitudes nB , mC ; it follows, that the fluid presses this vessel every where with a force proportional to the surface pressed and the height of the fluid above it.

532. Hence, if the line CE be drawn so as to meet the base AD produced in E ; and lines ag , bb , thro' the points a , b , parallel to the base; then because of the parallelism $Ca : Cb :: ag : bb$; it follows, that if DE expresses the pressure against the point D ; bb , ag , will express the pressure against the points b and a ; and the area of the triangle CDE will express the pressure against the line CD ; therefore the triangular prism, whose base is the triangle CED , and altitude the breadth of the side of the vessel, will express the pressure against that side.

533. The pressure against either side of a cubic vessel is just half the pressure against the bottom: For since the pressure against AD is as the rectangle of the same base and altitude of the triangle CDE ; and that against DC as the triangle itself; it follows that the pressure against the bottom is as the parallelepiped whose base is the rectangle made by the sides DC , DE , and altitude the breadth of the side, which is double the triangular prism, whose base is the triangle CDE , and of the same altitude. Therefore the pressure against the bottom is double the pressure against the side.

534. Since the pressure against the bottom is equal to the whole weight of the fluid, that against the side will be half that weight; and as there are four sides, the pressure against the four will be double the weight; and consequently, the sum of the pressures against the sides and bottom, is triple the weight of the fluid contained in the vessel.

535. If it was required to make the sides of the vessel of such a strength as to resist every where the pressure alike; let DE express the strength near the bottom, draw CE ; then the lines ag , bb , will represent the strengths at a , b .

It may be observed, that if a tube with a bottom fixed

to it, be filled with so much of a fluid as to make it burst in order to find its strength, altho' the greatest pressure is near the bottom, yet it will not burst there but somewhat higher; because the bottom keeps the sides together, and prevents them from giving way.

THEOREM.

536. *Fluids press every where according to their perpendicular altitudes, whatever their quantities are, or figures of the vessels in which they are contained. Fig. 4.*

Let ABEF be an upright cylindric or quadrangular vessel, with a tube of any length fixed to it: I say, that the bottom AF will be pressed with a force equal to the weight of a column of the fluid, whose base is equal to that of the vessel and altitude, to that of the vessel and tube together.

For suppose the fluid contained in the vessel ABEF to be divided into columns, whose bases are equal to that of the tube; then because the column LN endeavours to descend by its weight, so as to become level with the adjacent ones, these by reaction will endeavour to ascend to the same height, and so will press the surface BE with a force equal to the weight of the fluid contained in the tube CLD; these columns will, by the same reason, press the adjoining ones in the same manner, and they the surface BE. Therefore the surface BE is pressed upwards by a force equal to the weight of a column of the fluid of the same base, and whose altitude is equal to that of the tube CL. But because reaction is equal to action, the fluid cannot press the surface BE without pressing at the same time the base AF with the same force; and since the base AF supports likewise the weight of the fluid ABEF, it follows, that the base AF is pressed by a force equal to the weight of a column of the fluid of the same base, and whose altitude is equal to that of the vessel and tube together.

This may likewise be proved in this manner. Suppose the vessel to be continued to the height rs, of the tube;

tube; then because the fluid contained in the part $BrsE$ presses as much downwards on the plane BE as the fluid contained in $ABEF$ presses it upwards; and the base AF will be pressed by the whole column $ArsE$: Now if the weight of the fluid on the base $BCDE$ be taken away, and the resistance of that plane be taken instead of it, it is evident, that the fluid contained in $ABEF$ presses upwards against BE , with a force equal to that weight; but it cannot press upwards, without pressing at the same time the base AF downwards with the same force; to which adding the weight of the fluid contained in $ABEF$, the base AF will be pressed by a force equal to the weight of a column equal to $ArsF$.

This proposition has been confirmed by experiments; for if a vessel made so as to open like a pair of bellows, and a tube of any diameter be fixed to it; by pouring water into this tube, it will raise the upper base, loaded with a weight, nearly equal to that of a column of water of the same base, and an altitude equal to that of the vessel and tube.

Fig. 5. If the vessel is like a funnel, as $ABICD$, the pressure will still be equal to the weight of a column of the fluid which has the same base and altitude; because the column Ln answering to the tube, presses the bottom, and the adjacent columns with its whole weight, and by reaction they press the bottom with the same force, as well as all the rest of the columns; that is, in the same manner as if the vessel were of the same bigness $ArsD$ quite up to the top.

Fig. 6. If the vessel is wider above than below as $ABCD$, the base AD will only be pressed by the column $ArsD$ above it, and the rest of the fluid will be supported by the sides AB , DC , of the vessel.

Fig. 7. If the vessel was oblique, as $AbcD$, and of any figure whatever, the same thing will be true; that is, the base AD will be pressed by a column of the fluid whose base is equal to the base AD , and altitude to that of the vessel.

In general, of whatever figure a surface pressed

may be, the pressure will always be as the column of the fluid which presses it, and whose height is the depth of the fluid.

It seems to be surprising that a small quantity of a fluid should raise a very great weight; but when we consider what a prodigious weight a power applied to a machine made of wheels or screws may raise, there is no reason to doubt that a fluid applied in a proper manner may produce the same effect.

Fig. 1. 537. Hence, the bottom RS of the base of a syphon is every where pressed by a column of the fluid of an equal base, and the same altitude FL of the fluid above it, and the upper part E is likewise pressed upwards with a force equal to a column of the fluid, of the same base and altitude EL.

Therefore, when pipes are made to conduct water from one place to another, regard must be had to the height of the head above it; otherwise, they may be made too weak, and break, which would be attended with unnecessary expences.

Fig. 8. 538. If a rectangular surface ABCD, perpendicular to the horizon, be pressed by water, such as the gates of docks or sluices; the pressure against any horizontal line *ac*, will be as the rectangle made by that line and the depth *aA* under water; and the pressure against *bd* as the rectangle made by that line and the height *bA*. Therefore the pressure against the whole surface will be equal to a triangular prism of water whose base is equal to the surface ABCD, and altitude half the depth AD of the water. Consequently, this pressure is expressed by the solid $\frac{1}{2}AB \times AD$.

If $AB = 20$ feet, and $AD = 12$, that solid will contain 1440 cubic feet: Now as a cubic foot of common water weighs 62.5 pounds, it is evident, that the plane is pressed by a weight of 1440×62.5 , or 90000 pounds.

Fig. 9. 539. If the surface be oblique, such as is represented by ABCD, *An* being the surface of the water; by drawing *la*, *mb*, *nB*, perpendicular to *An*, and

and ac , bd , Be , perpendicular to AB , each equal to the respective depth; then it is evident, that the prism of water, whose base is the triangle ABe , and altitude the breadth of the plane $ABCD$, will express the pressure of the water against that plane; and any part of that prism, answering to the base $caBe$, that against the plane $aBCq$.

P R O B L E M.

540. Let AD be the breadth of a river or canal; to find the position of the gates AE , ED , such as to resist the pressure of the water with the greatest force possible. Fig. 10.

On AD as a diameter, describe a semi-circumference of a circle, draw the radius CB perpendicular to AD ; produce DE so as to meet the circumference in M , and join AM . Now as the pressure against DE or AE is as that line DE ; and as it becomes greater, its force decreases and the pressure increases; it is evident that the strength of the gate is reciprocally as the line ED ; but by the similarity of triangles, we have $DE : DC :: DA : DM$; and since DC and DA are constant, DM will express the strength of the gate AE or ED ; and because the strength of timber of the same diameter decreases, as it increases in length; the strength of the gates, in respect to that of the timber and the pressure together, will be expressed by DM^2 .

Again, the force with which the gate AE resists the pressure of DE at E , is equivalent to the forces EM and AM , the former parallel, and the latter perpendicular to DE . Therefore, the force with which one gate resists the pressure of another, is as AM . Consequently, $AM \times DM^2$ ought to be the greatest possible.

Hence, if $AD = a$, $AM = x$; then will $DM^2 = aa - xx$, and $AM \times DM^2 = aax - x^3$ which will be the greatest of all, when $3xx = aa$, by art. 245; for if the line $2AD$ ($2a$) be divided equally and inequally, so as the product of the rectangle $a+x$ by $a-x$, be multiplied by the middle part x , be the greatest possible, we shall find the value above of x . Consequently, AD (a) is to AM

($a\sqrt{3}$) as the radius 100000 is to the sine 57735 of the angle ADM, which will be found to be 35 degrees and 16 minutes.

What has been said here is also true in roofs of buildings; for the demonstration is the same in both cases: And by the similarity of the triangles DCE, DMA, we get $CE^2 = \frac{1}{3} aa$ or $CE = \frac{1}{\sqrt{3}} a$; this agrees very nearly with the common rule of making roofs, viz. $CE = \frac{1}{3} a$, or CE one third of AD.

T H E O R E M.

541. *The quantity of pressure against any surface is equal to the solid whose base is the surface, and altitude the distance of its center of gravity, from the surface of the fluid.* Fig. 8.

I. Let the rectangular surface ABCD be perpendicular to the horizon, its center of gravity will be (*art.* 420) in the middle of the line which bisects the sides AD and BC: Therefore $\frac{1}{2}AD$ will be its distance from the surface, and $\frac{1}{2}AB \times AD$; the quantity of the pressure, which is the (*art.* 538) same as before.

Fig. 9. II. Let the surface be oblique as AC; its center of gravity will be in the perpendicular which bisects AB; and hence its distance from the surface An of the fluid, will be equal to half of Bn or Be; and calling D the breadth of the surface, we shall have (*art.* 538) $\frac{1}{2}Be \times AB \times D$ for the pressure.

III. Let aB be the side of an oblique surface, whose breadth is D, its center of gravity will be in the perpendicular which bisects aB ; let the point b bisect aB ; then bm , or its equal bd , will be the distance of this center from the surface, and (*art.* 538) $\frac{1}{2}b d \times aB \times D$, the quantity of pressure against that surface.

542. Hence, if a surface or number of surfaces of any kind be considered as equally ponderous in every equal part, and as divided into innumerable small parts, suspended by lines, drawn from their centers perpendicular to any horizontal plane; it is manifest, that if every part be multiplied by its distance from that plane, the sum of the products will be equal to the product

duct of the whole surface or surfaces, multiplied by the distance of their common center of gravity from the said plane, by *art.* 422; this equality will subsist even if the said lines be perpendicular to any plane, though not parallel to the horizon.

543. Hence, because the center of gravity of a sphere, right cylinder, prism, or cube, is the same as the common center of gravity of their surfaces; the pressures upon these solids immersed in a fluid, will be equal to the product of their surfaces, multiplied by the distance of their center of gravity from the surface of the fluid.

544. If the center of gravity of any body immersed in a fluid, be in the same vertical line with the common center of gravity of its surface, that solid will either remain at rest, or descend in that vertical line; and it will never be at rest, unless these centers are in the same vertical line.

S E C T. V.

BODIES immersed in FLUIDS, and their SPECIFIC GRAVITIES.

T H E O R E M.

545. *The weights or gravities of bodies are in the compound ratio of their magnitudes and densities.*

For the denser bodies are the more matter they contain, and the greater their magnitudes are, the more matter they also contain. Therefore, as the weights or gravities of bodies are equal to the quantities of matter they contain, the weights of bodies are in the compound ratios of their magnitudes and densities.

546. Hence if two bodies are equal in magnitude, their weights are as their densities; and if their densities are equal, they are as their magnitudes: But if their weights are equal, their densities are reciprocally as their magnitudes.

547. Since the specific gravity of bodies, are as the weights of equal magnitudes, they are therefore as their densities,

densities, Consequently, whatever is said in respect to the densities of bodies, may also be understood in regard to their specific gravities.

THEOREM.

548. *If a body be immersed in a fluid, it will remain in equilibrio, swim, or sink, according as its specific gravity is equal, less, or greater, than that of the fluid.*

For a body immersed in a fluid will press downwards by the force of gravity; but as it cannot descend without moving as much of the fluid out of its place as is equal to it in bulk; it is evident, that it will be pressed upwards by the weight of as much of the fluid as it removes from its place; and therefore, if the specific gravity of the body is equal to that of the fluid, it will remain at rest in any depth; since the fluid cannot be compressed by supposition, and the force of gravity is the same, whether the same space be filled by the fluid or by the body.

If the specific gravity of the body be greater than that of the fluid, it must of consequence descend, by the excess of its weight above that of the fluid; and if it be lighter, it will be pressed upwards by a greater force than that whereby it endeavours to descend; and therefore it will ascend with a force equal to the difference between the specific gravities of the fluid, and its own.

549. Hence, if the specific gravity of a body immersed, be less than that of the fluid; it will sink so much only as to fill up a space, which contains as much of the fluid as is equal in weight to it. For if the body weighs just as much as the fluid which takes up the same space, all the ambient parts of the fluid will be pressed in the same manner as if that space was filled with the fluid, and consequently there will be an equilibrio between the fluid and the body.

550. Whence, the magnitude of the whole body is to the magnitude of the part immersed, as the specific gravity of the fluid is to that of the body; since when the weights of two bodies are equal, their densities, or

which

which is the same, their (*art.* 547) specific gravities, are reciprocally as (*art.* 546) their magnitudes.

Fig. 12. 551. If a body *b* specifically heavier than the fluid be immersed and supported by a power *P*, this power will be equal to the excess of the specific gravity of the body above that of the fluid; since the body would descend by a force (*art.* 548) equal to that excess.

552. Consequently, if the weight of a body immersed in a fluid be subtracted from its weight in the air, the difference will be the weight of an equal bulk of the fluid, and therefore is to the weight in the air, as the specific gravity of the fluid is to that of the body.

Examp. Let a piece of copper weigh 9 ounces in the air, and 8 in water; then the difference one is to 9 as the specific gravity of water is to that of copper.

553. Hence, if *a* expresses the specific gravity of the body *A*, whose weight in water is *B*: and *c* the specific gravity of water; then by the last corollary we have

$$A - B : A :: c : a = \frac{cA}{A - B}.$$

Consequently, the specific gravities of bodies will be as their weights in the air directly, and their loss in the same fluid inversely; since the specific gravity *c* of the fluid is constant, and *A - B* expresses the loss of the weight in the fluid.

554. If the same body be weighed in several fluids, and is specifically heavier; then because $a = \frac{cA}{A - B}$

gives $\frac{a \times A - B}{A} = c$, or only $A - B = \frac{aA}{c}$, since *a*, *A*, are here constant: It is evident, that the specific gravities of these fluids are to one another as the weights of the body lost in the fluids.

R E M A R K.

From what has been said in regard to the specific gravities of bodies, solid or fluid, in the three last corollaries; the specific gravities of bodies may easily be had by means of an hydrostatic ballance; but as they are found already by several curious persons, especially Mr.

Cotes,

Cotes, in his hydrostatic lectures, I shall insert part of his tables, which, in my opinion, are the best, leaving to those who have an opportunity the pleasure of trying them.

SPECIFIC GRAVITY.

Fine gold	19640	Dry boxwood	1030
Standard gold	18888	Sea water	1030
Quick-silver	14000	Common water	1000
Lead	11325	Dry oak	0925
Fine silver	11091	Dry ash	0800
Standard silver	10535	Dry maple	0755
Copper	9000	Dry elm	0600
Cast brass	8000	Dry fir	0550
Gun metal	8784	Cork	0240
Steel	7850	Air	0001
Iron	7645	Sand	1520
Cast iron	7425	Light earth	1984
Tin	7320	Loom	2160
Stone of mean gravity	2500	Marble	2700
Brick	2000	Pitch	1150

N. B. I have taken so much of *Mr. Cotes's* table as may be most useful to an engineer, and have added the specific gravities of gun metal, cast iron, and the different kinds of earth. It has been found by experiments, that a cubic foot of common water weighs 1000 ounces averdupois; and therefore, the numbers in the table express not only the specific gravities of the bodies mentioned therein, but likewise the exact weights of a cubic foot of each; and by proportion the weight of a greater or less bulk of these bodies may be found.

This table serves likewise to find the magnitudes of irregular bodies; for instance, suppose a piece of dry elm weighs 20 pounds, or 320 ounces, say as 600 ounces is to 320 ounces, so is a cubic foot or 1728 cubic inches to the magnitude required, which is 921.6 cubic inches.

P R O B.

P R O B L E M.

555. *The specific gravities of two ingredients of which a mixture is made being given, as likewise the specific gravity of the mixture, to find the quantity of each ingredient contained in that mixture.*

Let a , b , express the specific gravities of the ingredients, c the weight of the compound, d its specific gravity; then if z expresses the weight of the ingredient whose specific gravity is a , $c-z$ will be that whose specific gravity is b ; and $\frac{z}{a}$, $\frac{c-z}{b}$, will be (art. 553)

the weights lost in the fluid, and $\frac{c}{d}$ that of the mixture which must be equal to the sum of the former. Hence,
 $\frac{z}{a} + \frac{c-z}{b} = \frac{c}{d}$; and therefore $z = \frac{ac}{d} + \frac{b-d}{b-a}$.

Examp. Let a mixture of copper and tin, weigh 112 pounds, whose specific gravity is found to be 8784, that of tin 7320, and that of copper 9000; then will $a=7320$, $b=9000$, $d=8784$, $c=112$, and $b-d=216$, $d-a=1464$, and $b-a=1680$. Therefore $z = \frac{7320 \times 112 \times 216}{8784 \times 1680} = 12$, will be the number of pounds of tin in the mixture, and $112-12=100$, the number of pounds of copper.

556. It is also manifest, that having the specific gravities of the parts, that of the mixture may be found; for because $z = \frac{ac}{d} \times \frac{b-d}{b-a}$, we get $bdz - adz = abc -$

acd , by multiplication; and $d = \frac{abc}{bz - az + ac}$. Consequently, having the weights of the parts and their specific gravity, the weight of the compound being given, its specific gravity is given likewise.

557. If the weights of the ingredients are called m , n ; then will $m=z$, $n=c-z$; and the equation $\frac{z}{a} + \frac{c-z}{b} = \frac{c}{d}$

$\frac{c-z}{b} = \frac{c}{d}$, gives $\frac{m}{a} + \frac{n}{b} = \frac{c}{d}$. Hence we get $a =$

$$\frac{bdm}{bc-dn} \text{ or, } \frac{c}{d} - \frac{n}{b} : m :: 1 : a.$$

This equation furnishes a means to find the specific gravity of bodies, specifically lighter than the fluid: For if to this body another is joined much heavier, whose specific gravity is known; then finding the weight of the compound, as likewise its specific gravity, and dividing the weight m , of the lighter body by the

(art. 546) difference $\frac{c}{d} - \frac{n}{b}$, between the magnitudes of

the compound and the heavier body, the quotient will give the specific gravity of the lighter body.

Examp. If a piece of elm weighs 15 grains, having joined to it a piece of copper of 18 grains; let the specific gravity of the compound be $\frac{11}{9}$; then will $m = 15$;

(art. 557) $\frac{c}{d} = 27$; and $\frac{n}{b} = 2$.

Whence $\frac{c}{d} - \frac{n}{b} = 25$, and $a = \frac{15}{25} = 0.6$ equal to the specific gravity of elm; supposing that of copper to be 9.

S E C T. VI.

HYDRAULICS.

HYDRAULIC is the science which considers the motion of fluids, and the forces with which they act upon machines.

T H E O R E M.

558. *The vecocities of water in any two vertical sections AB, CD, of a river or canal EF, are reciprocally as the areas of these sections.* Fig. 13.

For the quantity of water running through one in any given time, is constantly equal to that which runs through the other in the same time, otherwise the water would rise in one place and fall in another, which is contrary

trary to experience: And because the quantity of water which runs thro' a section in a given time, is equal to a column whose base is equal to the section, and altitude to the space run over by a body in the same time with the velocity of the water: It follows, that the quantity of water run through a section is in the compound ratio of the section and velocity; therefore, since the quantities of water run through any two sections, at the same time, are equal, and the altitudes of equal solids (*art. 204*) reciprocally proportional to their bases, and these altitudes being as the velocities, the velocities are reciprocally as the sections.

559. If a expresses the height through which a falling body may acquire the velocity of the water running thro' the section AB , and b that thro' which a falling body may acquire the velocity of the water in the section CD ; then because the velocities of falling bodies are (*art. 450*) as the squares of the heights thro' which they fall, we have $a:b::CD^2:AB^2$. Consequently, the heights through which falling bodies may acquire the velocities of running water, are reciprocally as the squares of the sections.

REMARK.

If the velocity in every part of the same section was the same, it would be an easy matter to find the quantity of water discharged by a river or canal in any given time; but experience shews the contrary; for at the sides, or near the bottom, it will run much slower than in the middle of the stream, as being continually retarded by friction; in order therefore to find the true velocity by which the discharge is to be estimated; that at the sides, middle, and near the bottom, must be found, then the mean between them will be the velocity required.

If for example, the velocity in the middle of a stream is such as to carry a body over a space of 12 feet in a minute, near the bottom 8, and 6 at the sides; then the sum 26 of these three numbers, divided by 3, gives $8\frac{2}{3}$ feet run through by mean velocity in a minute.

The

The best way of finding the velocity of a stream is, to let an empty bottle corked, swim in it; and to make it go near the bottom, some water must be let into it, so as to be very little specifically heavier than water, and then to tie a piece of cork to it, which will keep it from sinking, and at the same time shew the velocity of the water.

T H E O R E M.

560. *If a right cylinder ABCD be constantly kept full, the velocity with which the water runs out at the bottom, will be equal to that which a falling body would acquire by descending through the height of the cylinder. Fig. 14.*

For since water descends by the force of gravity in the same manner as all other bodies, and there is no other obstacle to retard its motion by supposition; it is evident that the velocity with which the water begins to run out, increases gradually, till as much is run out, as the vessel contains, and then becomes uniform and equal to that which a body would acquire by falling thro' the height AB of the cylinder.

It may be observed that the cylinder must not be too small, otherwise the adhesion of the water to the sides will retard its motion, as appears in capillary tubes; besides, this rule takes place only after the upper surface BC is descended to the bottom.

561. It is manifest, that the velocity of water at the bottom is constant and uniform; for whatever the velocity of any section, when descended to the bottom, may be; that of any other which descends thro' the same height, must be the same.

Fig. 15. 562. If the cylinder be oblique, such as EFGH, the velocity of the water at the bottom, will be the same as that which runs through a right cylinder of the same base EH, and altitude FL; for we have proved (*art.* 536) that the base EH is pressed by the same weight as if the cylinder was upright, and therefore the generating force being the same, the velocity produced must be so too.

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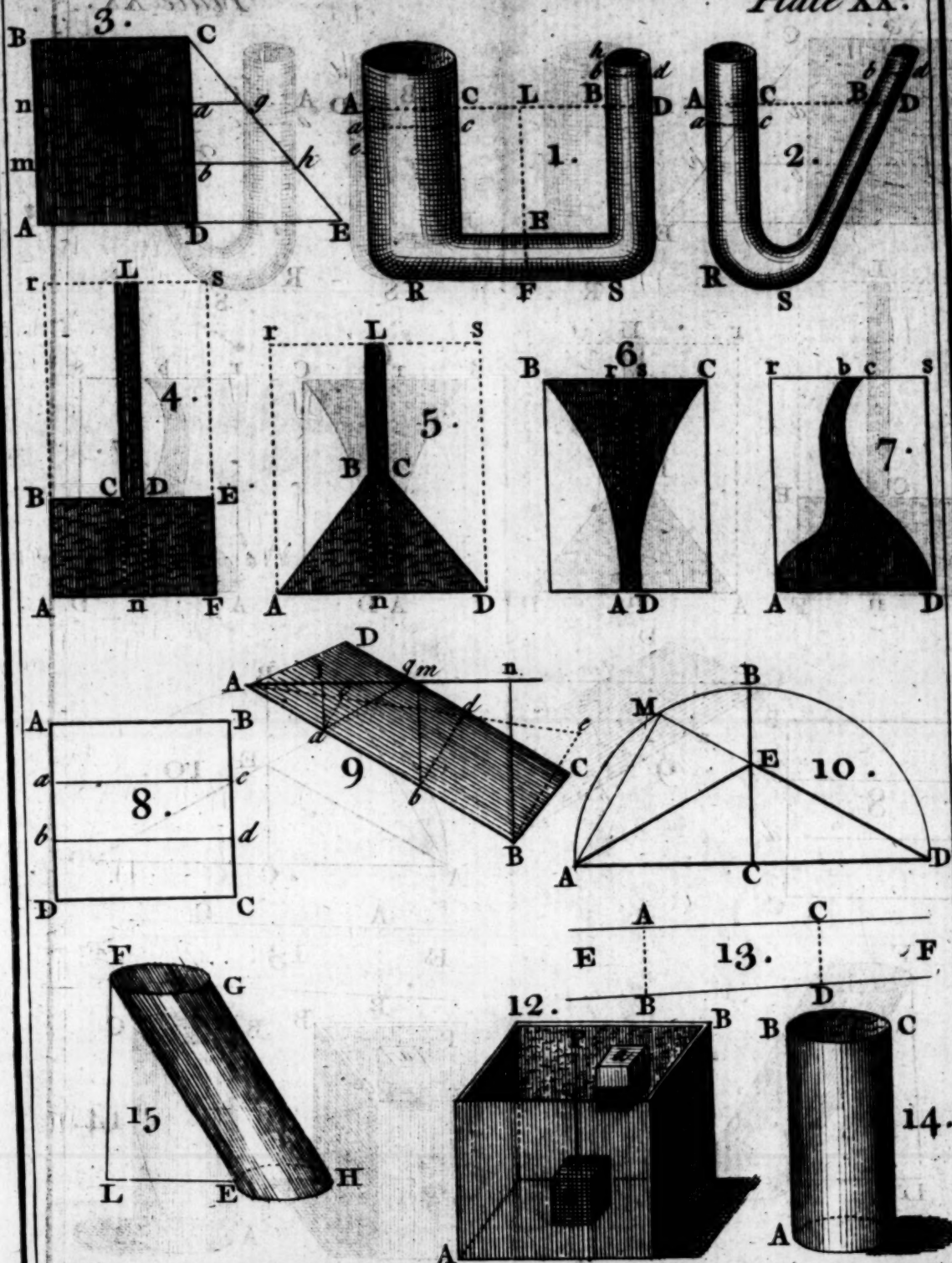
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563. The velocity of the water at the base, when the cylinder is constantly kept full, is double the velocity when no fresh supply of water is let in above.

P R O B L E M.

564. To find the velocity with which water runs out of an orifice LM made in the bottom of an upright vessel ABCD. Fig. 15.

Case I. If the vessel be constantly kept full, in AB take the point K such as twice the square of the surface EF, minus the square of the orifice is to the square of the surface EF as the height AB is to AK; that is, if EF^2 and LM express the areas, and $2EF^2 - LM^2 : EF^2 :: AB : AK$; then the fourth term AK will be the height thro' which a falling body would acquire the velocity with which the water runs out at the orifice.

Case II. If the vessel is not supplied with water whilst that which is in it runs out; then if $3EF^2 - LM^2 : EF^2 :: AB : AK$; the height AK will be such as a body falling through it, would acquire the velocity with which the water runs out at the orifice.

As the demonstration of this problem depends on the method of fluxions, we shall refer the reader to our Treatise on that subject, soon to be published.

565. Hence, if the orifice is but small in comparison to the surface EF of the vessel, then $2EF^2$ may be taken for $2EF^2 - LM^2$ in the first case, without any sensible error in practice; and therefore $2EF^2 : EF^2 :: AB : AK = \frac{1}{2} AB$, which agrees with Sir Isaac Newton's rule *. And when the orifice is equal to the surface EF; that is, when the vessel has no bottom, then the proportion in the first case becomes $EF^2 : EF^2 :: AB : AK = AB$, which agrees with what has been demonstrated in art. 560.

And in the second case the proportion becomes $2EF^2 : EF^2 :: AB : AK = \frac{1}{2} AB$, which agrees with what has been said in art. 563.

* Principia, prop. 36, book II.

566. It is therefore manifest, that the velocity of water running out of an orifice in the bottom of a vessel kept constantly full, can never be less than that acquired by a falling body through half the height of the water above the orifice, nor greater than that acquired by falling through the whole height.

567. If x expresses the height through which a body falls from a state of rest, in a second of time, and $AK = b$; then the velocities acquired by a body falling through the heights x, b , will be (*art.* 450) as their square roots. And because a body describes a space double, (*art.* 451) uniformly with the velocity acquired in the fall at the same time; it is evident that the velocity $\sqrt{x}$ is to the velocity $\sqrt{b}$, as the space $2x$ described uniformly with the velocity acquired in a second,

is to the space $\frac{2x\sqrt{b}}{\sqrt{x}}$ or $2\sqrt{xb}$ described uniformly

in the same time with the velocity acquired by falling thro' the height AK .

568. Since the quantity of water discharged through an orifice in any given time, is in the compound ratio of the velocity and the surface of the orifice; if we call $\odot$ the orifice LM ; then will $2 \odot \sqrt{xb}$ express the quantity of water discharged in a second.

569. Because a body falls through a space of 16.1 feet, or 193.2 inches, from a state of rest, in a second, in the latitude of *London*; if we suppose $x = 193.2$, we shall have $2 \odot \sqrt{193.2b}$, for the quantity of water discharged in a second, expressed in inches, supposing the height AK , and the orifice LM , to be expressed in inches.

570. Since a body falls through a space of 15 feet, or 180 inches, *French* measure, in a second, in the latitude of *Paris*, as has been found by experiments, if we suppose $x = 180$, then will $2 \odot \sqrt{180b}$ express the quantity of water discharged in a second in *French* measure.

EXAM-

EXAMPLE I.

571. Let the height of the water in a reservoir, above the orifice, be 13 feet *French* measure, and the diameter of the orifice .25 parts of an inch; then because the orifice is but small, in comparison to the breadth of the vessel, we may suppose (*art.* 565) $b = 6.5$ feet, or $b = 78$ inches. Therefore $180b = 14040$, or $\sqrt{180b} = 118.4$; and since the diameter of the orifice is .25 inches, its area $\odot$ will be $= .049$. Hence, if these values are substituted into (*art.* 570) $2 \odot \sqrt{180b}$ we shall have 11.6 cubic inches of water discharged in a second, or $11.6 \times 60 = 696$ inches in a minute.

Mr. *Mariotte* has found * by several experiments, that a reservoir of such dimensions as we have supposed here, discharged 14 pints of water in a minute, or because he supposes 35 pints in a cubic foot, 691.2 cubic inches; which comes very near to what we have found by the theory. The experiments will always give less than the true quantity, if the water is not let run till it has acquired its due velocity.

EXAMPLE II.

572. Let the height of the water in a reservoir be 25 inches *English* measure, and the orifice a square inch; by supposing $b = 12.5$, and $\odot = 1$; we shall have (*art.* 569) $2 \odot \sqrt{193.2b} = 98$ cubic inches nearly.

Dr. *Desaguliers* has found † by repeated experiments, as he says, that a vessel filled with water to the height of 25 inches, discharged 5.2 tuns in an hour, through a square orifice of an inch. Now because a tun of wine measure contains 58212 cubic inches, 5.2 makes 302702.4 cubic inches, which being divided by 3600, the number of seconds in an hour, gives about 84 cubic inches of water discharged in a second, which is less by 14 inches than what the theory gives.

* *Regles des jets d'eau*, p. 486.

† Page 128, vol. II.

As the Dr. has not given the breadth of the vessel, nor mentioned what measure he used, it is hard to know from whence this great difference arises. But as our theory agrees with Mr. *Mariotte's* experiments, which are known to have been made with great care and exactness, I am satisfied that the rule we have laid down will always answer nearly such experiments as are made with accuracy.

If the orifice be made in the side of the vessel, the rule laid down in the last problem will hold good in this case likewise, provided the mean velocity be taken; for the water will run faster near the bottom than above, when the orifice is but small, the height of the fall may be taken from the center; but when it is large, the mean velocity must be found in the manner shewn by authors.

If the vessel be oblique, and its width taken horizontally every where the same, the foregoing rule will still hold good. For it has been proved (*art. 536*) that the bottom of such vessels, or any horizontal section, is pressed in the same manner as an upright one of the same base and altitude; and therefore the generating force being the same, the velocities generated must be equal.

T H E O R E M.

573. *If to an upright vessel ABCD, there be joined a vertical tube EF, and in FE produced upwards, the point K be taken in such a manner, that a body falling through the height KE, acquires the velocity with which the water runs into the tube, KF will be the height through which a falling body acquires the velocity of the water at the orifice F. Fig. 16.*

For it has been proved (*art. 560*) that the velocity acquired by the water in the descent through a tube is equal to that acquired by a body falling thro' the same height. Therefore the velocity of the water, acquired by descending from the vessel through the tube, is equal to that of a body falling thro' the height KE.

T H E O-

THEOREM.

574. *If to an upright vessel ABCD, an horizontal tube DEFG is joined; the velocity with which the water runs out at the orifice GF, will be as the square root of the difference between the height EK through which a falling body acquires the mean velocity with which the water enters the pipe, and a part of the tube DG, which is in a constant ratio to the whole length. Fig. 17.*

For it is evident that the water would run out of the orifice DE, in the same manner as if it were in the bottom, if the water in the tube DF did not retard its motion; but this obstacle is in proportion to the quantity of water in that tube, or because it is of an equal bigness, as the length DG; and since the motion of a body falling through KE, is every where retarded, the velocity at the orifice ED will, of course, be less than if there was no obstacle; and therefore the velocity at the orifice GF, is as the square root of the difference between the height EK, and a part of the length DG of the tube, which is in a constant ratio of the whole.

The resistance of the motion in the horizontal pipe is not only owing to the quantity of water in it, but likewise to the friction of the water against its sides; but as this resistance is likewise proportional to the length of the pipe, the whole resistance which retards the motion of the water, will still be proportional to the length of the pipe.

In the conduct of water from one place to another, where the pipes ascend and descend, wind and turn about, it is impossible to give any certain rule for the quantities of water discharged, we shall add some observations, useful in the conduct of water. In pipes which rise and fall, the air that gets in with the water, rises to the highest parts and there it remains, whereby the motion of the water is much retarded. To remedy this, small vertical pipes are inserted in those places, with cocks to let out the air when the motion of the water begins to diminish. It happens likewise, that when the water in the reservoir is muddy it leaves dirt in the
T 3 pipes,

pipes, and sometimes choaks them up; in such a case bellowses are used to force them, by which they are cleared again.

THEOREM.

575. *If a tube EFGHI be fixed to the bottom of a reservoir ABCD, and the point K be so taken in the line EF produced upwards, that a body falling through the height KE may acquire the velocity with which water runs into the tube; then the water will run out of the orifice I with a velocity equal to that of a body falling through the height of the point K above the orifice. Fig. 16.*

For the velocity of the water in the lowest part F, will be equal to that acquired by a body falling through KF; and supposing the horizontal part FG of the pipe to be but short, it will rise in the vertical part GH, with the velocity at F, and lose as much of its velocity in the ascent (*art. 453*) as a body would acquire by falling through the height HG. If therefore the horizontal line LH be drawn, the water will lose just as much of its velocity in the ascent through GH, as it acquired by descending through LF, and consequently will run out of the orifice H, with the velocity acquired by a body falling through the height KL of the point K above the orifice.

576. Hence it is evident, that if the vertical part GH is but short, and the orifice upwards, the water will rise nearly to the same height with the point K; for the resistance of the air, and the friction in the pipe, will diminish the height of the water by some small matter.

Those who made experiments on jets of water have neglected the dimensions of the cisterns or reservoirs, so that it is not possible to know how much the rise will be less than the fall; they all imagined that it should be as high as the surface of the water, because they took this height to be that through which a falling body acquired the velocity of the water; experience has indeed shewn, that it does not rise so high; but the defect was attributed to the resistance of the air and that of the tube; though the difference between the
rise

rise and fall of water is not so much, considering the true height of the fall; yet it is certain that the resistance in the tube, and that of the air, diminishes the rise in high jets, very considerably. It is a common thing to fix a short brass tube to the orifice, of about an inch long, and of a less diameter than the other pipe, or only a thin plate of brass with an opening; this last is esteemed the best, on account, as it is said, that the resistance is almost nothing through this orifice, and to make it rise to its full height the jets are somewhat inclined from the vertical, so as hardly to be perceptible by the eye, that the water above may not fall back upon that which rises, and hinder its motion.

The pipe through which the water descends, is made larger than that through which it ascends; but the proportion is not certain, and is chiefly guessed at by the writers of experiments.

Mr. *Belidor* will have it * that the velocity with which the water runs out at the orifice I, is to be expressed by the difference of the square root of the descent KF, and that of the ascent GI; contrary to all other authors, and to what we have proved in the last theorem; and he thinks, (*art. 1211*) that when the ascending part of the tube GH is the $\frac{1}{3}$ of KF, the water will run out of the orifice with a greater velocity than if it were of any other; but this is as erroneous as the former; for it is plain, even according to his own principles, that the less the height GH is, the greater that velocity will be, and therefore when GH is very small, or nothing, the velocity at the orifice will be the greatest of all, since it is then equal to that acquired by a body falling through the height KF.

BODIES moved by FLUIDS.

THEOREM.

577. *If a plane surface be perpendicular to the stream of any fluid, the impression of the fluid on this plane will*

* *Architecture byd. vol. 2. book 4. art. 1225.*

be in the compound ratio of the square of the velocity and the surface pressed.

For with twice the velocity the impression of each particle will be double, and twice the number of particles will strike the plane in the same time, setting aside all impediments, and with a triple velocity, the impression of each particle will be triple, and three times the number of particles will strike the plane in the same time: Therefore the impression will be quadruple with twice the velocity, nintuple with thrice the velocity. By the same way of arguing, the impression will always be found to increase, in the same proportion as the squares of the velocities: And because the surface of the plane expresses the number of particles striking at the same time; it follows, that the sum of all the particles striking the plane in any given time, or the impression of the fluid on the plane, is in the compound ratio of the square of the velocity of the fluid, and the area of the plane.

578. Hence, if the surface be oblique to the direction of the stream, the absolute force of the particles is to the part striking the plane in a perpendicular direction, as the radius is to the sine of the incident angle, by art. 446 and the number of particles, which would strike the plane directly, is to the number of particles striking the plane obliquely, as the radius is to the line of the incident angle; it follows, that the square of the radius is to the square of the sine of the incident angle, as the absolute force of the fluid is to the part with which it strikes the oblique plane.

579. Because the absolute force of the fluid with which it would strike the plane in a perpendicular direction, is as the product of that plane and the square of the radius; the relative force of the fluid with which it strikes that plane in a direction oblique, will be as the product of that plane, and the square of the sine of the incident angle.

580. If a fluid strikes a plane which is in motion, in a perpendicular direction, it is evident that each particle strikes that plane with a velocity equal to the difference

rence

rence between (*art.* 465) the velocity of the fluid and that of the plane; and consequently the impression of the fluid on that plane, will be as the square of the difference between the velocities of the fluid and that of the plane.

P R O B L E M.

581. *To find the greatest momentum possible of a plane surface moved by a fluid.*

Let a express the constant velocity of the fluid x , the difference between the velocities of the fluid and that of the plane; then because xx , expresses (*art.* 580) the force of the fluid against that surface, and this force multiplied by the velocity $a-x$ of the surface gives $axx-xx^2$ for (*art.* 245) its momentum; it is evident that this expression $axx-xx^2$, ought to be the greatest possible, which is when (*art.* 249) $3x=2a$. Consequently, when the velocity $(a-x) = \frac{1}{3}a$ of the plane is one third of the velocity of the stream, it will have the greatest momentum possible.

582. Hence, because xx expresses the force of the fluid, which becomes $(xx=) \frac{4}{9}aa$, when it produces the greatest momentum possible. If there be a plane, such as the pallets of a wheel, which moves an engine; and this engine being charged with such a weight P , as will be just in equilibrio with the pressure of the water; it is evident that $P=xx$; and when the engine has the greatest momentum possible, we have $P=\frac{4}{9}aa$. Consequently, when the engine is first charged with a weight, so as to remain in equilibrio, and then with the $\frac{4}{9}$ th of that weight only, it will produce the greatest effect possible.

When a plane surface is moved by a fluid, and turns about an axis, such as the pallets of a water wheel, there is a point in that plane, which being struck by the whole force of the fluid, will produce the same effect, as if every part of that plane were struck by its respective force. This point is called the Center of Percussion: I have given a general method for finding this center

center in my Treatise of Conic-sections, page 159; but as it depends on the method of fluxions, we shall not attempt to give a demonstration here, but refer the reader to that book.

S E C T. VII.

P N E U M A T I C S.

AS air has a considerable share in the effects of pumps, syringes, fire-engines, barometers, and especially in the effects of gunpowder, it is necessary to explain here, in a few propositions, its principal properties, as far as they are useful, leaving those which are more curious than necessary in machines, to other authors, our chief design being to write for the practical reader.

T H E O R E M.

583. *If all the particles of air have the same elastic force, the force of compression is as the density.*

For since the particles are all endued with the same elastic force, by supposition, the compressing force must be as the number of particles pressed; but equal bulks of air, whose densities are twice, three times, four times, greater than that of another of the same bulk, contain (*art. 546*) twice, thrice, four times the number of particles; and in the same manner it is proved, that the density is always proportional to the number of particles pressed. Therefore the force of compression is as the density.

584. Hence, since the quantities of matter contained in a body is as (*art. 545*) its density and magnitude conjointly; if the same quantity of air be contained in different spaces, the densities are reciprocally as the spaces in which they are contained; and since we have proved that the compressing force is as the density, it follows that the compressing forces are reciprocally as the spaces, which contain the same quantity of air.

Fig. 27. 585. Hence, if a cylinder ADHE is filled with

with air, and that quantity of air be reduced to the space $ACGE$, or to $ABFE$, the force which retains the air in the space $ADHE$, is to the force which retains it in the space $ACGE$, or $ABFE$, as the height AC or AB is to the height AD of the cylinder: For these spaces having the same base AE , are to each other as their altitudes.

586. If AK be perpendicular to AD , and with the asymptotes AD , AK , there be described an equilateral hyperbola LMN , by drawing any where the lines BL , CM , DN , perpendicular to AD ; then the compressing forces which reduce the same quantity of air into the several spaces $ADHE$, $ACGE$, $ABFE$, are as the corresponding perpendiculars DN , CM , BL . For because $AD : AB :: BL : DN$, by the (*art.* 351) property of the hyperbola; and AD is to AB (*art.* 583) reciprocally as the compressing forces at BF to the compressing force at DH ; we have BL to DN as the compressing force at BF is to the compressing force at DH , by equality of ratios.

587. Since the rarity of an homogeneous fluid is reciprocally as the density, and $AD : AB :: BL : DN$; by the above noted property of the hyperbola, the rarity of the same quantity of a fluid contained in $ABFE$ is to its rarity when contained in $ADHE$, as AB is to AD .

588. Since AB decreases as BL increases, it is evident that when AB becomes indefinitely small, BL becomes indefinitely great; and therefore an elastic fluid may be compressed ad infinitum, provided a sufficient force could be found to do it.

P R O B L E M.

589. *To find how much the air is rarified in a receiver, after any given number of turns.*

Let the capacity of the receiver be to that of the barrel of the pump, as c to 1; then because the air will be rarified by any one of the turns in the ratio of the space c to the space $1 + c$; that is, it will be rarified $\frac{1+c}{c}$

times

times in two turns, $\frac{1+c}{c}$ in three, and, in general, $\frac{1+c^n}{c}$ times in n turns.

If the capacity of the barrel be equal to that of the receiver; that is, if $c=1$, then will 2^n express the number of times that the air is rarified. If for example, $n=10$, then will $2^n=1024$. Hence, in ten turns the air is 1024 times rarified.

590. Hence, if it be required to find the number of turns, so as to rarify the air any certain number r of times; then will $\frac{1+c^n}{c} = r$; or by the noted property of logarithms, we have $nl \frac{1+c}{c} = lr$; or $nl \overline{1+c} - l.c = lr$. Hence $n = \frac{lr}{l \overline{1+c} - l.c}$.

Examp. Let $r=100$, $c=1$; then will $n = \frac{l 10}{l 2} = 6.6$; so that in six turns, and six-tenth part of a turn, the air will be 100 times rarified.

THEOREM.

591. *If a glass tube, of about three feet long, be filled with quicksilver, and its orifice covered with a finger; by inverting the tube, and immersing the end stopped with the finger in a vessel filled with quicksilver, upon the removal of the finger from the orifice, the quicksilver in the tube will descend till its weight is in equilibrio with that of the ambient air.*

For because the upper end of the tube is stopped, the air cannot press upon the quicksilver in the tube, but pressing the surface of that in the vessel, with a force equal to its own weight; and the quicksilver in the tube presses also that surface by its whole weight; it follows necessarily, that the weight of the quicksilver in the tube must be in equilibrio with that of the ambient air.

592. Hence it is manifest, that the weight of the quicksil-

quicksilver in the tube, above the surface of that in the vessel, is equal to the weight of a column of air of the same base with the tube, and whose altitude is equal to the height of the atmosphere.

593. The weight of the air varies at different times; for the quicksilver in the tube is found to rise and fall from the height of 31 inches to 28, which variation must be owing to the different pressures of the air on the surface of the quicksilver in the vessel, since the weight of the quicksilver remains the same.

N. B. It is upon this principle that the barometers and other weather-glasses are made: For a barometer is nothing else but a glass tube of about 32 inches long, inverted into a vessel filled with quicksilver, and the interval between 31 and 28 inches, from the surface of the quicksilver in the vessel, being divided into any number of equal parts, by which the rise and fall of the quicksilver in the tube, and, consequently, the different pressures of the air, are distinguished.

594. Hence, the weight of a column of air of any given base may be computed. For since a cubic foot of quicksilver weighs 14000 ounces, or 875 pounds, according to the tables, (*art.* 554) when the quicksilver stands at the height of 30 inches, or 2.5 feet; a column of quicksilver, whose base is a square foot, and its altitude 2.5 feet, will weigh 2.5×875 , or 2187.5 pounds; which is therefore also the weight of a column of air at that time, standing on a base of a square foot.

The surface of a middle-sized man is computed to be about 15 square feet; and therefore the weight of the air which presses such a surface will be 15×2187.5 , or 32812.5 pounds.

From whence it seems to be surprizing that we should not perceive this great pressure; but it must be considered, that we are every where pressed alike, by which it is felt no where; whereas if it was taken away in any part, it would be so great as we should be hardly able to bear it; for by laying a hand on the mouth of the air pump, in a few turns the pressure appears so great, that

that if more air was pumped out, it would break the hand.

THEOREM.

595. *If a tube be filled with water, or any other fluid, and immersed into a vessel full of the same fluid, with the open end downwards; the height to which that fluid will rise in the tube, will be to the height of the quicksilver in the barometer at that time, reciprocally as the specific gravity of the quicksilver is to the specific gravity of the fluid.*

For because both the fluid and the quicksilver in the tubes being in equilibrio with the same weight of the ambient air, their weights must be equal to each other; but the weights of bodies are in the compound ratio (*art.* 545) of their densities and magnitudes, and the columns of the fluid and the quicksilver, which have equal bases, are as their altitudes, and the specific gravities (*art.* 547) as the densities: It follows, that the height of the fluid in the tube is to the height of the quicksilver in the barometer, as the specific gravity of the quicksilver is to the specific gravity of the fluid.

596. Hence, since the specific gravity of water is to that of quicksilver as unity to 14; when the quicksilver stands at 30 inches high, if we say, $1 : 14 :: 30 : 420$, this fourth term will express the height to which water will rise in a tube at that time, expressed in inches; or the water will rise to the height of 35 feet at that time: But if the quicksilver stands at 28 inches, then the water will only rise to the height of 32.5 feet.

597. Hence, water can be raised no higher than about 34 feet, by means of a sucking pump; because this pump draws the air out of the pipe, by which the pressure of the ambient air forces the water to rise to such a height, as to be in equilibrio with that pressure, which has been proved to be about 34 feet at a medium; and therefore the water cannot rise higher. For which reason, if it be required to rise it to a greater height the forcing pump must be used.

PROBLEM.

598. *If a tube be filled with quicksilver, or any other incompressible*

compressible fluid whatsoever. all but a part ef , which remains filled with air; when the tube is inverted, and its orifice immersed in a vessel filled with the same fluid, it is required to find the height bc to which the fluid will rise.
Fig. 28.

Let bd be the height to which the fluid would rise, if no air was included in the tube; then the pressure of the atmosphere upon the surface of the vessel would be (art. 592) balanced by a column bd of the fluid; but here it is balanced by the weight of the column bc , and the pressure of the included air in ce downwards; the weight of the column bd is therefore equivalent to the weight bc , and to the elastic force of the air in ce . Whence, if the column bc be taken away, which is common to both, the weight of the fluid in cd will be equivalent to the elastic force of the air in ce ; but when the air is in its natural state, it ballances the weight of the fluid in bd ; and this weight is to that in cd , as bd to cd ; and by the elasticity of the air, the force, when confined in ef , is to the force when confined in ce (art. 585) as ce is to fe . Consequently, $bd:cd::ec:ef$, by equality of ratios.

If de be bisected in g , and we suppose $bd=b$, $ef=c$, ge or $gd=a$, and $cg=x$; then will $cd=x-a$, $ec=x+a$; and by substituting these values into the proportion above, we get $b:x-a::x+a:c$, or, $bc=xx-aa$; therefore $x=\sqrt{aa+bc}$.

Examp. I. Let the length be of the tube be 35 inches, and the height $bd=30$; then will $ef=5$, and $gd=2.5$; whence, by making $a=2.5$, $b=30$, $c=5$, we get $cg=\sqrt{6.25+150}=12.5$, and $gc-gd=12.5-2.5$; or, $cd=10$, and $bc=20$.

Examp. II. Let the height be of the tube be 36 inches; $bd=30$, and let the quantity of air ef be required so as the quicksilver may rise 18 inches; then because $de=6$, $gd=3$, $cd=30-18=12$, we have $a=3$, $x=15$, $b=30$; and therefore the equation $bc=xx-aa$, gives $c=\frac{xx-aa}{b}=7.2$ inches, for the height ef of the air required.

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Fig. 29. When the length of the tube is less than the height to which the fluid would rise; the same equation will still hold good. For if the length of the tube $be = 23$ inches, $ef = 2$, and the height bc to which the fluid will rise; then by supposing $bd = 30$, we get $bd - be = de = 7$, $ge = a = 3.5$, $b = 30$, and cg , or $x = \sqrt{12.25 + 60} = 8.5$; hence $cg + gd = ed = 12$, and $bc = 18$.

P R O B L E M.

549. *To find the altitude to which the quicksilver will rise when there is a given quantity of any other fluid in the same tube.*

It is evident, that the sum of the weights of the quicksilver and the fluid, is equal to the weight of the ambient air, or to the column of quicksilver, which is equal in weight to the air: If therefore the height of the column of quicksilver, equal to the column of the fluid, be subtracted from the column of quicksilver which ballances the weight of the air, the difference will be the height to which the quicksilver will rise in the tube, being charged with that quantity of the fluid.

If a expresses the height of the fluid, x that of quicksilver, whose weight is equal to that of the fluid, and b the height of the quicksilver, which ballances the pressure of the atmosphere, then will $b - x$ express the height of the quicksilver in the tube, and $a + b - x$ that of the quicksilver and fluid together.

Now if m is to n as the specific gravity of the quicksilver is to that of the fluid; then because these columns have the same base, they are as their altitudes; and since the weights are as (*art. 545*) the specific gravities and magnitudes conjointly, an will be the weight of the fluid, and mx that of the quicksilver, which is equal to it; and therefore $mx = an$ by supposition, or

$x = \frac{an}{m}$. This value being substituted into $b - x$, and

$b + a - x$, gives $b - \frac{na}{m}$, for the altitude of the quicksilver,

silver, and $a + b \frac{n}{m}$, for that of the quicksilver and fluid together.

Examp. Let there be water in the tube to the height of 28 inches, and the height of the quicksilver in the barometer 30 at the same time; then because m is to n as 14 is to unity, according to the table of specific gravity, we have $b = 30$, $a = 28$, $m = 14$, $n = 1$, $b \frac{n}{m} = 28$, for the height of the quicksilver, and $a + b \frac{n}{m} = 58$, for that of the water and quicksilver together.

SECTION VIII. PUMPS.

HAVING applied the principles laid down in every section to some useful subject, I cannot but chuse to do the same here, in regard to hydraulics; but because of its great extent it is impossible to enter into all the particulars in such a work as this, I shall be content to explain the nature and use of pumps only, as being the most simple, and at the same time the most useful, of all hydraulic machines.

There are three sorts, the sucking, the lifting, and the forcing pump; I shall give a description of each separately, after having said something concerning the several parts of which they are composed.

Fig. 30. The body ABCD of the pump is called the *Barrel*, which is a hollow cylinder, made of brass, copper, pot-metal, a composition of lead and copper, hard metal made of lead and tin, or hard lead; that is, the barrel is made of any substance which may be made smooth within; the wooden ones are made only for cheapness.

The sucking pipe EADF, is that which goes from the barrel down into the water, which is generally widened at the lower end, to give a free passage for the water to enter; there is fixed a thin iron plate at the entrance, full of holes, to prevent the dirt from getting into the pipe. This is made of any kind of hard or soft

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substance,

Fig. 29. When the length of the tube is less than the height to which the fluid would rise; the same equation will still hold good. For if the length of the tube $be=23$ inches, $ef=2$, and the height bc to which the fluid will rise; then by supposing $bd=30$, we get $bd-be=de=7$, $ge=a=3.5$, $b=30$, and cg , or $x=\sqrt{12.25+60}=8.5$; hence $cg+gd=ed=12$, and $bc=18$.

P R O B L E M.

549. *To find the altitude to which the quicksilver will rise when there is a given quantity of any other fluid in the same tube.*

It is evident, that the sum of the weights of the quicksilver and the fluid, is equal to the weight of the ambient air, or to the column of quicksilver, which is equal in weight to the air: If therefore the height of the column of quicksilver, equal to the column of the fluid, be subtracted from the column of quicksilver which ballances the weight of the air, the difference will be the height to which the quicksilver will rise in the tube, being charged with that quantity of the fluid.

If a expresses the height of the fluid, x that of quicksilver, whose weight is equal to that of the fluid, and b the height of the quicksilver, which ballances the pressure of the atmosphere, then will $b-x$ express the height of the quicksilver in the tube, and $a+b-x$ that of the quicksilver and fluid together.

Now if m is to n as the specific gravity of the quicksilver is to that of the fluid; then because these columns have the same base, they are as their altitudes; and since the weights are as (*art.* 545) the specific gravities and magnitudes conjointly, an will be the weight of the fluid, and mx that of the quicksilver, which is equal to it; and therefore $mx=an$ by supposition, or

$x=\frac{an}{m}$. This value being substituted into $b-x$, and

$b+a-x$, gives $b-\frac{na}{m}$, for the altitude of the quick-

silver,

fluid together. $a + b \frac{na}{m}$, for that of the quicksilver and fluid together.

Examp. Let there be water in the tube to the height of 28 inches, and the height of the quicksilver in the barometer 30 at the same time; then because m is to n as 14 is to unity, according to the table of specific gravity, we have $b = 30$, $a = 28$, $m = 14$, $n = 1$, $b \frac{na}{m} = 28$, for the height of the quicksilver, and $a + b \frac{na}{m} = 58$, for that of the water and quicksilver together.

S E C T. VIII. P U M P S.

HAVING applied the principles laid down in every section to some useful subject, I cannot but chuse to do the same here, in regard to hydraulics; but because of its great extent it is impossible to enter into all the particulars in such a work as this, I shall be content to explain the nature and use of pumps only, as being the most simple, and at the same time the most useful, of all hydraulic machines.

There are three sorts, the sucking, the lifting, and the forcing pump; I shall give a description of each separately, after having said something concerning the several parts of which they are composed.

Fig. 30. The body ABCD of the pump is called the *Barrel*, which is a hollow cylinder, made of brass, copper, pot-metal, a composition of lead and copper, hard metal made of lead and tin, or hard lead; that is, the barrel is made of any substance which may be made smooth within; the wooden ones are made only for cheapness.

The sucking pipe EADF, is that which goes from the barrel down into the water, which is generally widened at the lower end, to give a free passage for the water to enter; there is fixed a thin iron plate at the entrance, full of holes, to prevent the dirt from getting into the pipe. This is made of any kind of hard or soft

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substance,

substance, as wood, lead, or cast iron; because it requires not to be so smooth within as the barrel. These two cylinders are joined together at AD by two flat rings cast with them, called Bridles, screwed together with four screws, and to prevent the air or water from passing between, a ring of leather is put into the joint, which is here stained darker than the pipes: But when the pump is made of wood, the barrel and sucking pipe are both made of the same piece, or the sucking pipe is less than the barrel, and its end sharpened so as to be forced into the barrel, which is bound with an iron hoop to prevent its splitting. Common pumps are often made of a single leaden pipe.

The *Piston* is a solid *abc*, which moves in the barrel by means of a power applied to an iron rod *ef*, fixed to the handle *dd*; there are two sorts of pistons, the one hollow within, such as are represented in *fig. 30* and *31*; where the cavity is marked with two pricked lines which is called *Bucket*, of which there are several kinds; the most simple of all, and which is commonly used in ordinary pumps, is made of wood, in the form of a truncated cone, with its least base downwards, having an iron hoop or ring at each end, to prevent the wood from splitting; and about the upper part *bc* is fastened a piece of strong leather, reaching a little above the wood, by two rows of nails, to keep the bucket tight and close to the barrel, so as no water or air may pass between them.

The cavity of the bucket is covered by a *Valve g*, which is a round piece of leather, exceeding the diameter of the hole about an inch, and having a tong or flipe on one side, which being nailed on the bucket serves as a joint for the valve to turn about; and to prevent the leather from bending, a thin plate of iron is put over it of the same diameter, and sometimes another under it, of the bigness of the cavity of the bucket, all three rivetted together, or fastened with screws.

In sucking pumps, where the weight of the water
above

above the bucket is not very great, a piece of lead is only fastened to the leather by a screw, such as is represented here at *g*, to make the valve shut by its own weight. These buckets will be sufficient in ordinary pumps, being besides the cheapest; but in larger ones the best way of making them is as follows.

Fig. 33. Instead of the wooden cylinder, or rather truncated cone, let there be a brass one *m n*, with a brim or projection *k l* at the upper end of about half an inch broad, and so as to project the lower part by the thickness of strong leather, the part *x m* below this brim is covered with strong leather, fastened with an iron hoop *m* sunk into the leather so as not to project it.

There is a brass barr at each end, cast with the bucket, such as is seen at the upper end *k l*, with a square hole in the middle to receive the iron rod or handle of the piston. The valve of this bucket is a round piece of leather, nearly equal in diameter to that of the bucket, on each side of which are fastened two equal iron plates, with rivets or screws in the form of a segment of a circle, so as to leave the breadth of the barr *k l* of the leather uncovered, which serves as a joint to both parts to turn about; this part of the leather uncovered is laid on the barr *kl*, and fastened close to it by the iron piece *rs*, fig. 34, so that one end *t* being placed at *k*, the other *u* will be at *l*; the lower end *r* of this iron goes through the square holes of both bars, and is fastened underneath with a key, and to the upper end *s* is fixed the iron rod by which the piston is moved.

The two segments of the valve rise together by the pressure of the water coming through the bucket, and lean against the iron *rs*; and when the bucket ascends, fall back again and shut the cavity of the bucket.

Fig. 32. The solid piston *abc* used in forcing pumps, is a solid cylinder of wood or brass, somewhat less in the middle than at the ends, to receive a leather collar, which is nailed on it, when the cylinder is of wood, such as is represented here; so that the whole together

keeps the piston tight and close to the barrel, that no air or water may pass between them: The iron rod by which the piston is moved passes through it, and is fastened by two keys, one below and the other above it.

These sort of pistons are sometimes made of three brass cylinders which screw into each other; the middle one fills the barrel of the pump nearly, and the others are somewhat less; and to make the piston come close to the barrel, two leathern rings are inserted into the joints, by which no air or water can get between the piston and the barrel.

Authors have imagined various sorts of pistons, as well as different valves; but the most material ones, and which are commonly used, have been described; as for the rest, we shall refer the reader to Mr. *Belidor's* *Hydraulics*, or to Dr. *Desaguliers's* *Experimental Philosophy*.

Mr. *Belidor* proposes a new sort of valves, which is a brass circular plate, with an axis fixed to it, so as to divide the area into two segments, one being double the other; this plate is made very smooth, and ground so true to the bucket as to let pass no water or air between them.

When the bucket descends and the water presses against this valve, it finds less resistance on one side than on the other, by which the valve is forced to turn about its axis, with the larger segment upwards, to let pass the water; and when the piston ascends, the weight of the air or water above it presses more on the larger segment than on the smaller, by which the valve is shut again.

Fig. 31. In the lifting pump the piston is the same as in the sucking one, only inverted, and the valve is at the smaller end of the bucket.

At the end of the barrel where it is joined to the sucking, lifting, or forcing pipe, there is likewise a valve, made in the same manner as those in the piston, for which reason we shall not enter into any particulars about it, observing only that the leathern tong of the valve

valve is of a piece with the leathern ring between the joints of the pipes. When the barrel and sucking pipe are of a piece, the valve is fixed near the surface of the water, and sometimes under it; the box to which the valve is fixed is either a wooden or brass cylinder, and soldered to the pipe if it be of any kind of metal, and forced down when made of wood; in this latter case it will be convenient to have an iron ring fixed to it, in order to draw it out when occasion requires it.

Having explained the several parts of which pumps are composed, I shall now proceed to shew the manner of working them.

The S U C K I N G P U M P.

Fig. 30. When the piston descends to AD, the air contained between it and the valve *z*, being reduced into a less space, will be condensed (*art. 616*) and by its elastic force presses the valve *z* down, and forces the valve *g* of the bucket open, so as to rise above the piston; then by rising the piston the weight of the atmosphere presses the valve *g* down, so as no air can pass through the bucket, by which the air that remained between the bucket and the valve *z* will be rarified; and as that in the sucking pipe is denser than that above this valve, it acquires an elastic force, sufficient to raise the valve *z*, and so enters into the barrel, till the air in both is of the same density; then the atmosphere presses on the surface of the water in the well, causes the water to rise in the pipe; and when the piston descends, the air under the piston keeps the valve *z* shut, forces the valve *g* of the bucket open, and passes above the piston; and by rising the piston, the pressure of the ambient air keeps the valve *g* shut, and the air in the sucking pipe forces the valve *z* open; enters into the barrel, and the water in the well being pressed by the atmosphere rises again in the pipe: By continuing in this manner to move the piston, the water will rise into the barrel, and from thence into the cystem or reservoir KL, provided the height of the cystem above the sur-

face of the water in the well does not exceed 34 feet, or thereabouts: I have shewn (*art.* 626) that a column of quicksilver of 29.5 inches high, was in equilibrio, with the pressure of the atmosphere, and that water is about 14 times lighter than quicksilver; therefore a column of water whose height is $14 \times 29.5 = 413$ inches, or 34 feet and 5 inches, will counterpoise the weight of the atmosphere on the same base.

P R O B L E M.

594. *To find the height to which water will rise in the sucking pipe at every stroke of the piston, the length of the sucking pipe and that of the stroke being given.*

Suppose the piston to descend quite close to the valve z , divide the capacity of the barrel from AD to the end of the stroke, by the section of the sucking pipe, in order to reduce them both to the same diameter; and let c express the height of the column of water which is in equilibrio with the pressure of the atmosphere, a the length of the stroke of the piston, b the length of the sucking pipe, and x the height to which the water rises in the pipe at the first stroke.

When the piston is raised, the air in the pipe would be rarified into the space $a+b$, if the water did not rise in the pipe; but the water rising to the height x , the air will be reduced into the space $a+b-x$: Therefore its density before the stroke, will be to its density after the stroke, (*art.* 584) reciprocally as the space $a+b-x$ is to the space b ; but because the air was in equilibrio with the atmosphere before the stroke, it will be of the same density, and so its elastic force may be expressed by c :

Whence, if $a+b-x : b :: c : \frac{bc}{a+b-x}$, this fourth term will express the density or elastic force of the air after the stroke; which, together with the weight of the water x , are in equilibrio with the pressure of the atmosphere immediately after the stroke. Consequently,

$a+b-x$

$\frac{bc}{a+b-x} + x = c$, or $\frac{bc}{a+b-x} = c-x$, and $bc = ac + bc - ax - bx - cx + xx$, by multiplication. If $a+b+c=2n$, we shall have $xx - 2nx = -ac$, by reduction; and adding nn to both sides, the first will be a compleat square, whose root is $n-x = \sqrt{nn-ac}$, or $x = n - \sqrt{nn-ac}$.

The height to which water rises in any of the succeeding strokes will be found by substituting the difference between the height of the sucking pipe and that of the water in it, instead of b into the equation above.

595. Hence, if it be required to find the length of the sucking pipe, so as the water shall rise to any given height x ; the equation $ac - ax - bx - cx + xx = 0$, gives

$b = x + \frac{ac}{c+x}$; and if the length of a stroke was required to make the water rise to the given height x ; then will $a = \frac{bc}{x} + x - c - b$.

E X A M P L E.

Let $c=31$ feet, the height of the sucking pipe $b=28$, and the length of the stroke $a=8$; then will $a+b+c=2n=67$, or $n=33.5$; and substituting the values of a, b, c, n , into the equation $x = n - \sqrt{nn-ac}$, we shall have $x = 33.5 - 29.5 = 4$ feet nearly, for the height to which the water rises at the first stroke; and the space b to which the air is then reduced into, will be $28-4=24$ feet.

Now if we suppose $b=24$, and the rest as before, then will $a+b+c=2n=63$, or $n=31.5$, and $x=31.5-27.3=4.2$, for the height to which the water rises at the second stroke; and the air will be contained in the space of $24-4.2=19.8$. Whence, if $b=19.8$, then $a+b+c=2n=58.8$, or $n=29.4$, and $x=29.4-24.8=4.6$, for that height at the third stroke. In the same manner the height at the fourth stroke is found to be 5.1, and 8.7 at the fifth; when the air is reduced into a space of 1.4 feet, which being less than the height of

the stroke, the water must rise above the piston at the sixth, and from thence into the cystem.

It must be observed that this does not answer exactly in practice, because part of the air which enters into the barrel is forced down again, into the sucking pipe by the shutting of the valve; but, however, the difference is not very considerable.

These are the dimensions of a pump, in which Mr. *Belidor* says, that the water will never rise (Page 90) to a height of 10 feet; and he is surprized that Mr. *Mariotte* should propose it as a pattern for this kind of pumps: But I am more so, that he should not perceive the absurdity of this argument, and that Dr. *Desagulier* should fall into (Page 183) the same mistake.

In order to prove this he refers to *art.* 815, where he says, as the air cannot be pumped out to that degree so as none remains in the pump, the water will not quite rise 32 feet high; which he takes to be the height of the column of water that counterpoizes the pressure of the atmosphere: But how this may account for the water not rising above 9 feet and some inches, I leave the reader to judge.

It would be needless to expatiate further upon such an absurd argument, therefore I shall proceed on the subject.

The LIFTING PUMP.

Fig. 31. The barrel ABCD of this pump is supposed to be in the water; to the lower end AD is screwed a tube ADHG, wider at bottom than the barrel, to let the water run freely into the pump; the barrel is often made wider itself at the end, which saves expences; BEFC is a part of the lifting pipe, which goes up to the cystem or reservoir. When the piston is raised the water or air above it keeps the valve *g* shut, and forces the valve *z* open to enter into the lifting pipe; then when the piston descends, the water or air above the valve *z* presses it down and keeps it shut, whilst the water in the well forces open the valve *g* and enters above

bove the piston, which, by rising, lifts it up into the lifting pipe as before.

It is plain that this pump will raise water to any height, provided the moving power be sufficient. This power lifts up not only the water above the piston, but likewise that in the barrel below it; because when it rises it causes a vacuum between the surface of the water and the bucket, till the pressure of the outward air forces the water to rise in the pump; and therefore, as much of the pressure of the air is taken off in the tube, as is equivalent to the weight of the water which rises.

The FORCING PUMP.

Fig. 32. The piston *abc* of this pump is a solid cylinder, as we have observed before; the forcing pipe *GHID* is at first perpendicular to the barrel, and then it rises gradually up to the cystem or reservoir, or it turns up vertically, according as the cystem is near or at a distance from the well.

When the piston descends it compresses the air under it in the barrel, by which it passes into the forcing pipe, through the valve *g*; then when the piston rises it rarifies the air above the valve *z*, by which the air or water under it forces it open to enter into the barrel, whilst the valve *g* is kept shut by the pressure of the air in the forcing pipe; and by forcing the piston downwards, the air above the valve *z* is forced out through the valve *g*; and thus by continuing to move the piston, the water will rise into the barrel by the pressure of the atmosphere, and from thence it is forced up the forcing pipe into the reservoir.

This pump serves to raise water to any height as well as the former, and is preferable to it, inasmuch as the weight of the forcer diminishes the moving force; and it may be charged with such a weight, as that the power of forcing up the water be equal to that which raises the forcer, by which the motion of the machine becomes uniform, which is a great advantage, and cannot be had in lifting pumps, where the moving power is obliged

to

to lift up the weight of the piston and the tackle belonging to it, besides the water:

The MANNER of applying the MOVING POWER to the PISTON.

Fig. 30. In the sucking pump the power is applied to a short leaver which turns about an axis, the end of which, where the iron rod *ef* is fastened to, is but short, and seldom exceeds 6 inches; sometimes an iron bar is fixed to one end of the axis, with a weight at the end, and the leaver where the rod of the piston is fastened to, is perpendicular to the middle of this axis, as may be seen in most common pumps.

If it should be required to make the stroke of the piston pretty long, it would be proper to add a circular arc to the leaver, then the iron rod *ef* is fastened to it, by means of a chain of a few links to roll upon that arc, as may be seen in the fire-engine at *Chelsea*: But then, the piston must be heavy enough to descend by its own weight. As in common pumps the stroke of the piston is mostly very short, this method is seldom practised.

Fig. 31. In the lifting pump the iron rod of the piston is fixed to a rectangular iron frame, not unlike the wooden frames of some saws; and this frame is fixed with another iron rod to the cranks of an axis of a wheel, which is turned by water or horses; and to lose no power, the axis has generally two cranks bending contrary ways, in order to move two pistons, so that when one ascends, the other descends: The barrels of these two pistons are both joined to one lifting pipe, which therefore carries the water of both into the reservoir by a continued stream.

Fig. 32. In forcing pumps the iron rod of the forcer is fixed to a crank of an axis, in the same manner as in a lifting pump. There are various manners of applying the moving powers to the piston, that are used upon different occasions, which are impossible to be all described here, and may best be seen in authors who have wrote particularly upon this subject.

There

There is sometimes an air pipe added to the lifting or forcing pump, which is upright, and has a communication with the lifting or forcing pipe, of the size, or bigger than the barrel of the pump, and stopped close above with a cover screwed on it. When the air or water is forced into the lifting or forcing pipe, the air is condensed in the air pipe, and part of the water enters into it; and when the piston works, this air forces the water to rise in the pipe, by which it runs into the reservoir with a continued uniform motion; which is a great advantage to the machine, although it produces no more water than if there was none; but it must be observed, that this pipe is only used in a single pump; for when there are more than one, there is no occasion for it.

PROPORTIONS of the several parts of a PUMP.

Fig. 30. As the goodness of all machines depends on the exactness of the proportions between the parts, in respect to each other, I shall endeavour to explain here, in a few words, all that can be said of pumps: In order to which I will begin with those between the diameter of the barrel and the sucking pipe, which ought to be such, that the diameter of the cavity in the bucket, is exactly equal to that of the sucking pipe. For when this is the case, the water will rise every where with an equal velocity, which is allowed to be a great perfection; since if some parts are narrower than others, the velocity will there be so much the greater in proportion; and as the resistance of the pipe partly depends on the velocity, it must be made proportionably stronger there than in any other part: If the barrel be wider, the moving power must be greater; since the pressure of the air is always proportional to the surface pressed, that is, as the base of the piston.

The same thing must be observed with respect to the valve *z*; for wherever it be placed, its opening should always be the same as that of the bucket.

Mr. *Belidor* would have the valve *z* placed at the end of the barrel, in the manner represented in *fig. 30*; and
Dr.

Dr. Desagulier at the end of the sucking pipe, near or under the surface of the water. I must confess, that I cannot perceive any advantage in placing it in one part sooner than in another; for when it is at the end of the barrel, it may be easier come at, in case it wants mending, which it will oftener than if it was under water, where it is constantly kept wet, whereby it will be but seldom out of order; as to the goodness of the pump, it does not in the least depend on the situation of this valve, since it will discharge as much water one way as the other, at the same time.

The next thing to be considered is, the velocity with which the piston ought to move, in respect to that with which the water rises in the pipe; for if the piston moves faster than the water, there will be a vacancy left between them; and when the piston descends, it will not fetch so much water, as it should do; and if the velocity of the water is greater than that of the piston, it will be stopped; and therefore the pump will not furnish the quantity of water which it might, if the piston did move faster; for which reason we shall give the following

P R O B L E M.

To find the velocity of the water in the sucking pipe, at any given height.

As the water rises in the pipe by the pressure of the atmosphere on the surface of the water on the well, and this pressure is equivalent to the weight of a column of water of 34 feet height, as has been observed before; it is evident that this pressure is capable to generate a velocity equal to that of a body falling through that height; since, if the air is removed out of the pipe, the water would rise to that height; therefore the velocity of the water, at any height from the surface of the water in the well, is equal to the velocity of a body thrown upwards with a velocity acquired by falling through 34 feet, setting aside all obstacles or impediments; but the velocity of the body at any height,

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when thrown upwards, is equal to that which a body would acquire by (art. 453) falling through the difference, between the ascent and descent. It therefore expresses the difference between the height of the pump and 34 feet; then because a body falls through a space of 16.1 feet in a second, and with the velocity acquired during the fall, it would describe a space 32.2 double, uniformly in the same time; and since the velocities acquired are as the square roots of the heights, if $\sqrt{16.1} : 32.2 :: \sqrt{a} : 2\sqrt{16.1a}$, this fourth term will be the space described uniformly with the velocity of the water in a second of time.

Examp. Let the height of the pump from the surface of the water in the well to the cystem be 26 feet; then will $34 - 26 = 8 = a$; and therefore $2\sqrt{16.1a}$ gives 11.3 feet for the space described uniformly with the velocity of water in a second of time.

As the water meets with some resistance in the ascent of the tube, this velocity is rather more than the water really has at that height; for which reason the velocity of the piston should not be so much.

Mr. *Belidor* says, that he has seen no pump whose piston had above 4 *French* feet velocity in a second; if it be so, there is no danger that the water will not follow immediately the piston, at least when the pump is not higher than about 26 feet.

If the cavity of the bucket be greater than that of the valve z , or the section of the sucking pipe, the water will move slower in the barrel, in a reciprocal proportion of the section of the sucking pipe to that of the cavity of the bucket: whence, if D be the diameter of the sucking pipe, d that of the cavity in the bucket, by making $dd :: DD :: 2\sqrt{16.1a} : x$; the fourth term x will express the velocity of the piston such, that the water will immediately follow the piston.

Fig. 31. In the lifting pump the piston should move quite close up to the valve z , because the water which is not lifted through that valve, descends again with
the

the piston, is an additional weight to be lifted up by the moving power, and the pump does not discharge the quantity of water it ought to do.

Fig. 32. As to the piston in the forcing pipe, it ought not to descend lower than to about half the diameter GD of the forcing pipe; the valve *z* should be quite even with the bottom of that pipe; since whatever water remains above that valve is a weight to it, and prevents the water from rising so quick as it should do. And if the forcing pipe did incline a little downwards from the barrel to the valve *g*, so as the water which is not forced out, might run through that valve by its own weight, the pump would, in my opinion, be better than when that pipe makes a right angle with the barrel; since by that means all the water that rises above the valve *z*, would be discharged without any loss.

Mr. *Belidor* has given eight problems, pag. 92. relating to the perfection of pumps, formerly proposed by Mr. *Parent*, which were taken out of a manuscript containing a treatise on pumps, he wonders how it came not to be printed, and imagines that Mr. *Parent* wanted to make a secret of it. As I have considered with great attention the solutions given of them by Mr. *Belidor*, I cannot find that they are of any use, to the best of my judgment, which makes me believe that Mr. *Parent* found himself mistaken in proposing of them, suppressed the manuscript from which they were taken; but Mr. *Belidor* being fully bent upon giving, in his works, whatever seemed to be conducive to the perfection of machines, relying too much on Mr. *Parent*'s skill and knowledge, has without duly considering whether it was right or not, published these problems, which by the author himself were found to be of no consequence.

This is nearly all that can be said in regard to the pumps we have here been speaking of; there are, it is true, abundance of different manners in which they are adapted to machines, by which they are worked; but

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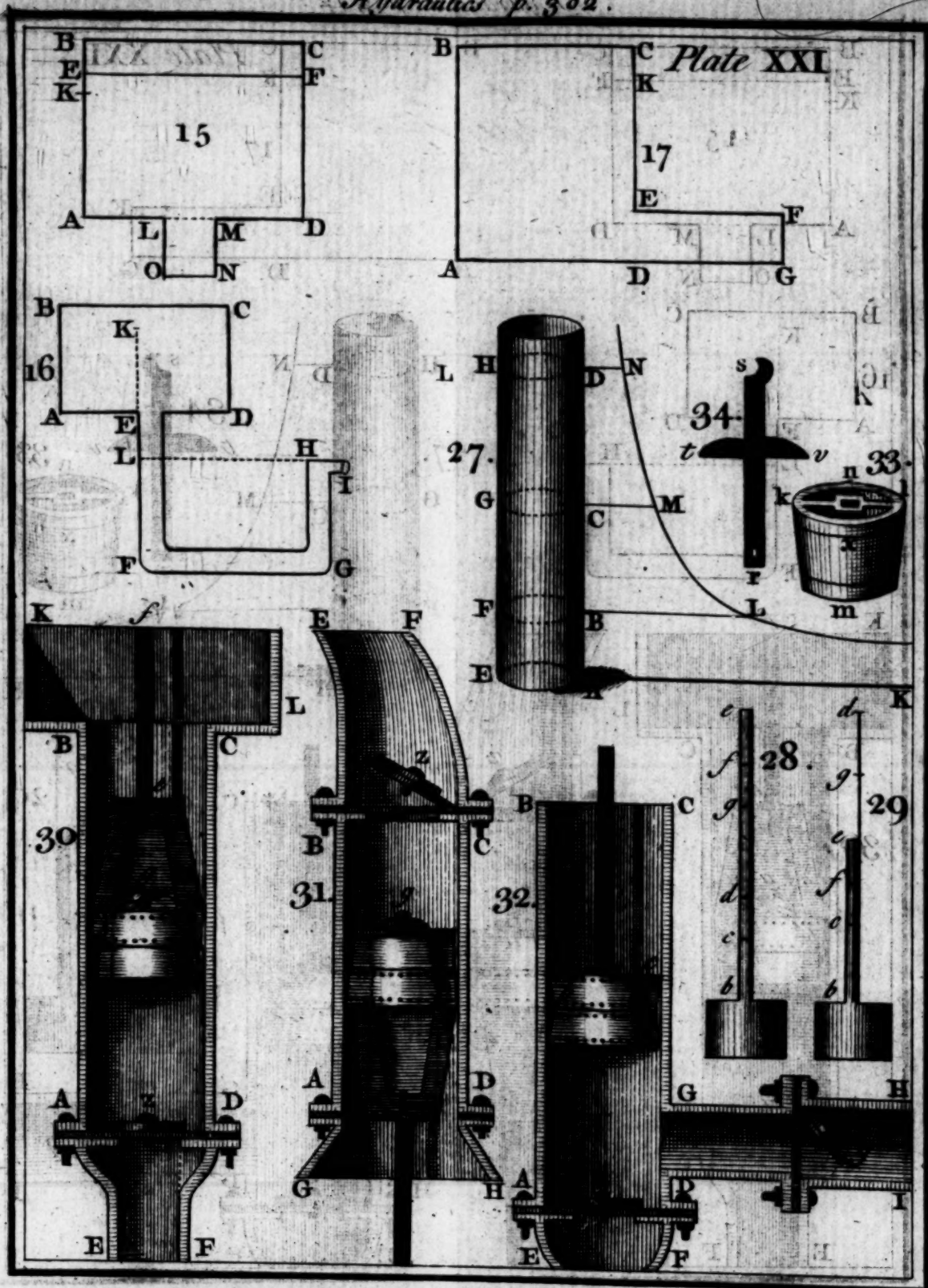
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as our design in this work is only to treat of such parts as are most useful to an engineer, or any other military gentleman, those whose curiosity incites them to study any part, in a more compleat manner, must consult those authors who wrote particularly on that subject, especially Mr. *Belidor's* works.

P. S. In perusing the memoirs of the royal academy of Sciences at *Paris*, of the year 1736, I found a treatise on pumps, wrote by Mr. *Pitot*, wherein he endeavours to prove, that in many pumps the water will not rise up to the cistern, though it be not 34 feet high; and in an example, where he supposes the stroke of the piston 2 feet, and the sucking pipe 14 long, that the water will not rise above 8 feet; as I have already taken notice of this mistake in page 302, I shall only add, that both Mr. *Belidor* and *Pitot* take the exhausting of the air out of an air pump and that out of a water pump to be the same thing; they do not consider, that when the air is exhausted out of the first, there is nothing that supplies its place; whereas in the latter, the water rises as the air is rarified, till it is again of the same density with the outward air; and therefore no reason can be assigned, why the water should rise no higher than these gentlemen would have it: If any such thing happens in practice, as they seem to insinuate, it must be owing to some imperfection of the pump; and not to what they have imagined. Mr. *Pitot* gives likewise the solutions of Mr. *Parent's* 8 problems; but as his principles are not better grounded than those given by Mr. *Belidor*, I have no reason to contradict what has been said in page 302, on that account.

2 M O I X A

P E R

PERSPECTIVE.

DEFINITIONS.

Fig. 1.

THE appearance *mn* of an object *MN*, seen by a spectator's eye placed at *O*, thro' a transparent plane *ABDC*, is called the *perspective* of that object.

2. The transparent plane *ABCD*, the *picture*.

3. *EFQ*, in which the object is placed, the *Ground plane*.

4. The intersection *CD* of the ground plane and the picture, the *ground line*.

5. If a plane passes thro' the eye *O* parallel to the ground plane, its intersection *AB*, with the picture, is called the *Horizontal line*.

6. If a plane passes thro' the eye *O* parallel to the picture, its intersection *EF*, with the ground plane, the *directing line*.

7. The point *O*, where the eye is placed, the *point of sight*.

8. If a line *OG* be drawn in the plane *OAB* parallel to any line *MN* in the ground plane, the point *G*, where it meets the horizontal line *AB*, the *vanishing point* of that line.

9. If a line *OP*, be drawn, in the plane *OEF*, parallel to the perspective *mn* of a line *MN*, in the ground plane, the point *P*, where it meets the directing line *EF*, the *directing point* of that line.

10. If a line be drawn, from any given point, perpendicular to any other line, this line is called the *distance* of that point from this line.

A X I O M S.

1. If from any point, two lines are drawn to the extremities of another line, these three lines will be in the same plane.

2. If a line be drawn thro' any point, parallel to any other line, these two lines will be in the same plane.

THE

T H E O R E M.

Art. 1. *Any line MN placed in the ground plane, and its perspective mn, will when produced, if necessary, meet the ground line in the same point L.*

For the points m, n, are in the rays, O N, O M, by defin. 1. and so mn is in the plane of the triangle ONM, by ax. 1. it is likewise in the plane of the picture; they will therefore meet, when produced, in the intersection CD of these two planes.

T H E O R E M.

2. *The perspective mn of any line MN in the ground plane, when produced, will meet the vanishing point G of that line.*

The line OG being parallel to MN, by definit. 8, will be in the plane OGNM by ax. 2, as well as mn, and they will meet, when produced, which therefore must be in the intersection AB, of the plane OAB and the picture.

T H E O R E M.

3. *Any original NM in the ground plane, when produced, will meet its directing point P.*

As the line OP is parallel to the perspective nm, by defin. 9. it is in the plane of the triangle ONM, by ax. 2. the lines OP, NM will therefore meet in the intersection EF of the plane OEF, and the ground plane:
C O R.

4. Hence, all lines, which are parallel to each other but not to the ground line, have the same vanishing point, since there can but one line be drawn thro' the point of sight parallel to them all.

C O R.

5. All lines parallel to the ground line, have their perspectives parallel to the same line: For if the directing point P moves along the directing line EF till OP becomes parallel to EF; then as the perspectives mn, are always parallel to OP, by definition 9; they will all become parallel to the same line EF or CD in this case, and so parallel to each other.

X

C O R.

C O R.

6. All parallel lines in planes parallel to the picture, have their perspectives parallel; since the perspectives will be parallel to their originals in this case by defin. 1.

C O R.

7. All lines which have the same directing point P, have their perspectives parallel, since they are all parallel to the same line OP, by defin. 9.

C O R.

8. All parallel planes which intersect the picture, have the same horizontal line; since there can but one plane pass thro' the point of sight, which is parallel to them all.

C O R.

9. If the picture turns about the ground line CD, till it coincides with the ground plane; as in fig. 2. it is evident, that the lines CD, AB, EF; the lines GL, OP, and NM, GO, remain parallel as before; the rays ON, OM intersect also the line LG, in the same points n, m. For the triangles OGN, nLN, remain similar to the triangle OPN, and the triangles OGM, mLM, to the triangle OPM; the proportion between the original NM and its perspective mn remaining the same, the perspective must also be the same.

N. B. The distance of the eye from the horizontal line AB, is always equal to the distance of the directing line EF from the ground line; for LGOP, is a parallelogram by def. 5, 9.

P R O B L E M I.

10. To find the perspective of a line NM in the ground plane. Fig. 2.

Let O be the point of sight, AB the horizontal line, CD the ground line; produce the line NM till it meets the ground line in L, from the point of sight O draw OG parallel to NL, and let G be the vanishing point of NM; then the lines drawn from the point of sight O to the extremities M, N, will determine the perspective nm in the line LG, by art. 1, 2. The perspective n of any point N, may also be found, by drawing from

from that point and that of sight O, two lines NR, OH parallel to each other, so as to meet the ground and horizontal line in R, H: Then the intersection of the line RH, and NO drawn from the point N and the point of sight, will be the perspective required.

P R O B L E M II.

11. *To find the perspective p of any right lined plane figure P. Plate II.*

AB represents the horizontal line, CD the ground line, O the point of sight, as in the former figures: OG, OH, are drawn parallel to the sides NL, LM; so that G, H, are the vanishing points of the lines LN, LM, and of their parallels RM, RN in the third figure by defin. 8. Now if the sides of the originals are produced till they meet the ground line, and from thence lines are drawn to their respective vanishing points, they will give the perspective required by art. 1.

When some of the sides of the originals do not meet the ground line when produced or are at an inconvenient distance, as NM in fig. 4. If lines are drawn from the point of sight O, to the extremities N, M, of that line, they will intersect the adjacent sides of the perspective in the points required n, m. The fifth figure is a large square divided into smaller ones, the vanishing points G, H, in the horizontal line AB, of the perpendicular sides RN, SM, to the ground line and of the diagonal NS, are taken at pleasure; and the perspective will vary and diminish more or less, as the horizontal line and these vanishing points are taken nearer or farther from the original.

P R O B L E M III.

12. *To find the perspective of a curvelined plane figure. Plate III. Fig. 7, 8.*

This is done by finding several points in the perspective, and joining them very neatly by hand. Thus, if a square LMNR, fig. 7. be described about a circle, and the side LR touches the ground line; draw the

diagonal RM, and thro' its intersections with the circumference lines perpendicular to the ground line, whose vanishing point is G and H that of the diagonal. Having drawn lines from the intersections of the perpendiculars and the ground line to the vanishing point G, and through the intersections, of RH and these lines, parallel to the ground line, they will give four points in the perspective and four lines that touches it.

As many more points as you please may be found by dividing the circumscribed square into any number of small ones, or by article 10.

In the eighth figure, the directing point P in the directing line EF, is taken at pleasure, and the circumference is divided into any number of parts, from which lines are drawn to the point P, and thro' the intersections of these lines and the ground line CD, other lines are drawn parallel to PO.

Then to find the perspective m of any point M, draw OM, which will intersect the perspective Nm of MN in the point required: in the same manner all the rest of the points are found; since all lines which have the same directing point have their perspectives parallel by art. 7, and are also in the lines drawn from their extremities to the point of sight.

The same perspective may likewise be found by drawing a great number of lines from the circumference, parallel to the line which joins the point of sight and the vanishing point, and from their intersections with the ground line other lines to the vanishing point, the rest may be completed as above.

P R O B L E M IV.

13. *To find the perspective of a regular solid.* Plate III. Fig. 9, 10.

Find the perspective of the base by the second and third problems, and erect perpendiculars at every angle if the figure be a prism: then if one of the angles L, fig. 9, touches the ground line CD, make Ll equal to its altitude, and draw from the point l two
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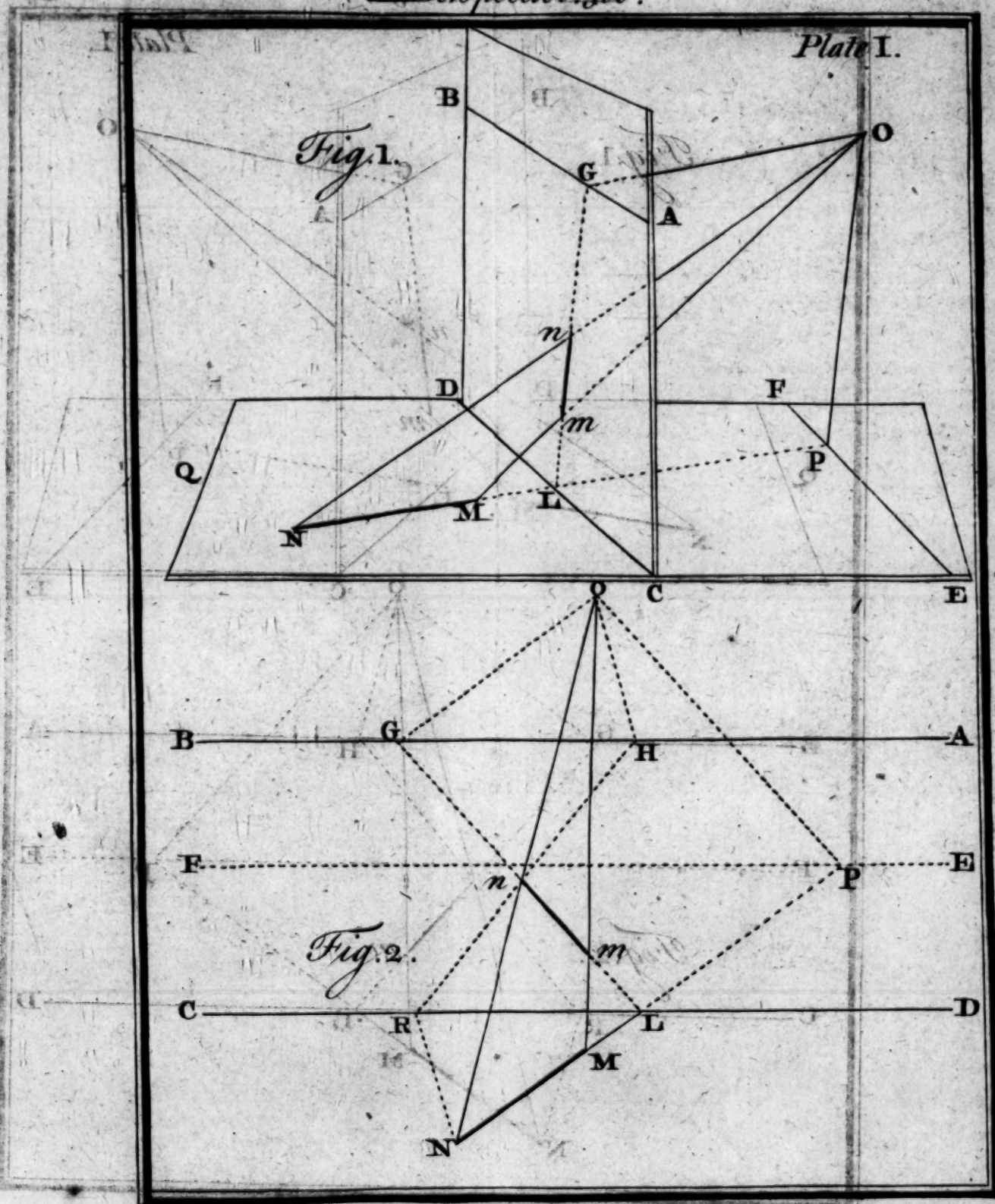
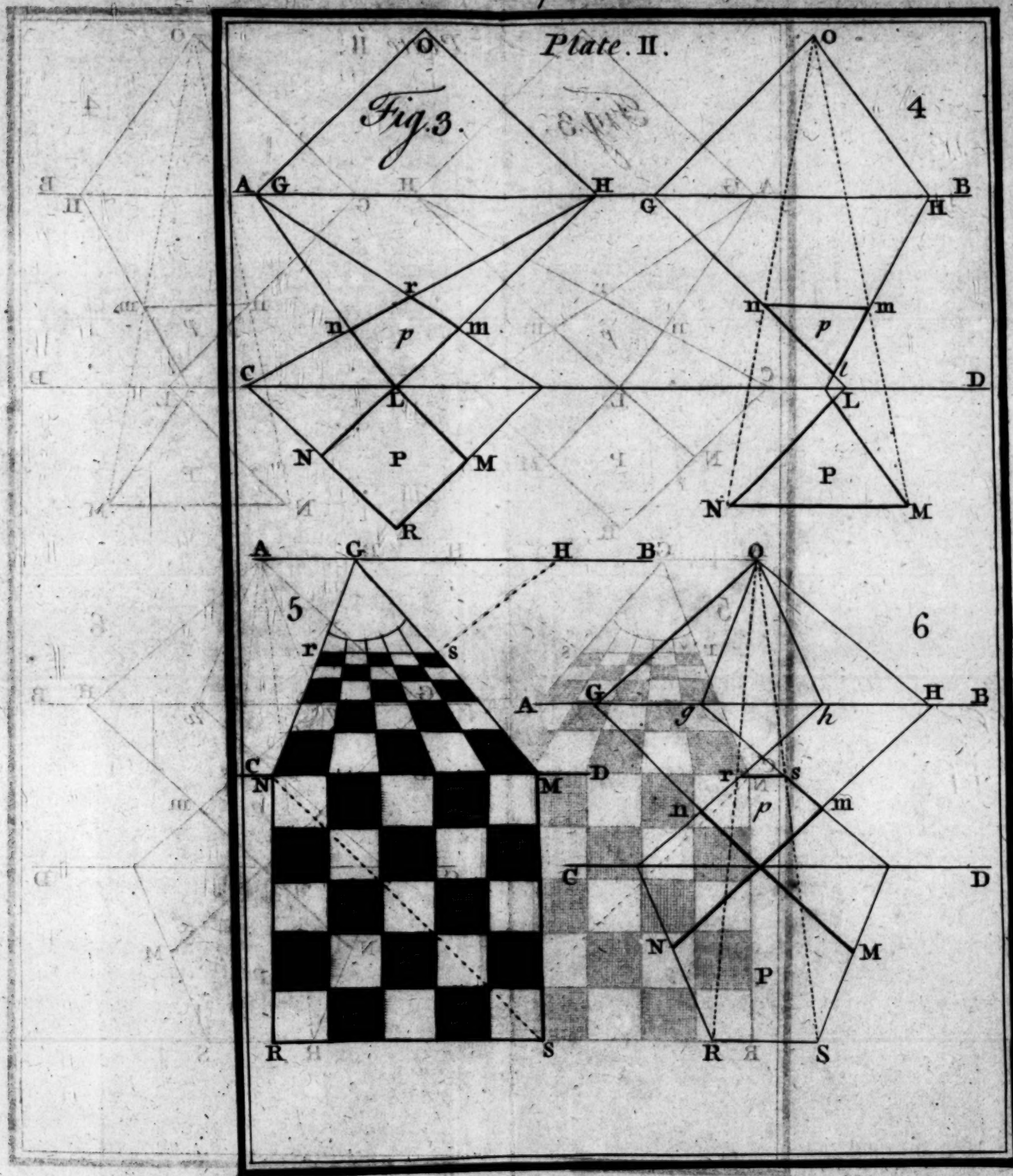


Plate. II.

Fig. 3.



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lines to the vanishing points G, H ; which here fall out of the figure of the lines ML, LN ; these lines will intersect the perpendiculars in m, n , and the lines drawn from the points m, n , to the vanishing points H, G , will determine the point r .

For the plane passing thro' the point l , is parallel to the ground plane by supposition; these two planes have therefore the same ground line by art. 8. and all the parallel lines in parallel planes have the same vanishing point by art. 4. The tenth figure is a square pyramid, AG the horizontal line, G the vanishing point of the sides, perpendicular to the ground line LG , the line passing thro' the center of the base and vanishing point, Ll the altitude of the pyramid, and the line lG intersects the axis in the vertex K of the perspective.

For the plane passing thro' the point l parallel to the ground plane, will have the same horizontal line AG , by art. 8. and the vertex K is in that plane.

Plate IV. The eleventh figure is a prism whose base is a regular exagon, AB the horizontal, and KP the ground line, A, B the vanishing points of the lines GD, FC , and their parallels; Kk the altitude of the prism, the line Bk determines the side bg , of the upper base, Ab the side bc , and gf, cd , are parallel to the horizontal line, which determine the points f, d ; and the lines Af, dB , the sides fe, de .

For the plane passing through the point k parallel to the ground plane, has the same horizontal line AB as the ground plane, and the parallel lines in parallel planes have the same vanishing point, by art. 4.

Fig. 12. is a frustum, whose opposite bases are parallel and similar concentric exagons, $A B$, the horizontal line, A, B , the vanishing points of the parallel lines which meet the horizontal line; the perspective of the bases being found as in prob. 2; erect perpendiculars at all the angles of the least base, and make Kk equal to the height of the frustum, the line kB will, with two perpendiculars, determine the side ab , the other sides of the upper base are found as above; then the respec-

tive

tive angles of the opposite bases being joined, gives the perspective required.

Or if the lower base be found, and Kk be produced to the height of the compleat pyramid, and finding the perspective of that height, by drawing lines from the angles of the lower base to that point, the upper base of the perspective may be found as before, without finding its perspective in the ground plane, and raising perpendiculars upon its angles.

Plate V. The house A and rows of trees marked with *o*, are drawn into perspective, by supposing CD the ground line, KV the horizontal line, as usual; the point of sight is placed above the middle of the house, V is the vanishing point of the perpendiculars to the ground line, K that of the parallels *m*1, *n*2, *p*3, *q*4; by laying a ruler to the point K, and to the points 1, 2, 3, 4, successively, you mark in the line *q*V the several points where the parallels cross that line; these parallels being drawn, will intersect the lines drawn to the point V, where the trees and house are to stand; if thro' these points perpendiculars are drawn, and those near the ground line made equal to the respective heights, the lines drawn from these points to the point V, will determine the heights of the trees and the house.

Plate VI. represents the perspective of a room with several regular solids in it. AB is the horizontal line, CD the ground line, O the point of sight, V the vanishing point of the perpendiculars to the ground line; the perspectives of the solids being found as before as well as that of the ground plane; then the several heights of the sides are found; the lower and upper parts are terminated by lines drawn to the vanishing point V: the upper and lower part of the door have the same vanishing point *a*, taken at pleasure. The wainscoting and ornaments of the cieling might also be drawn as well as any furniture, but we shall omit them here.

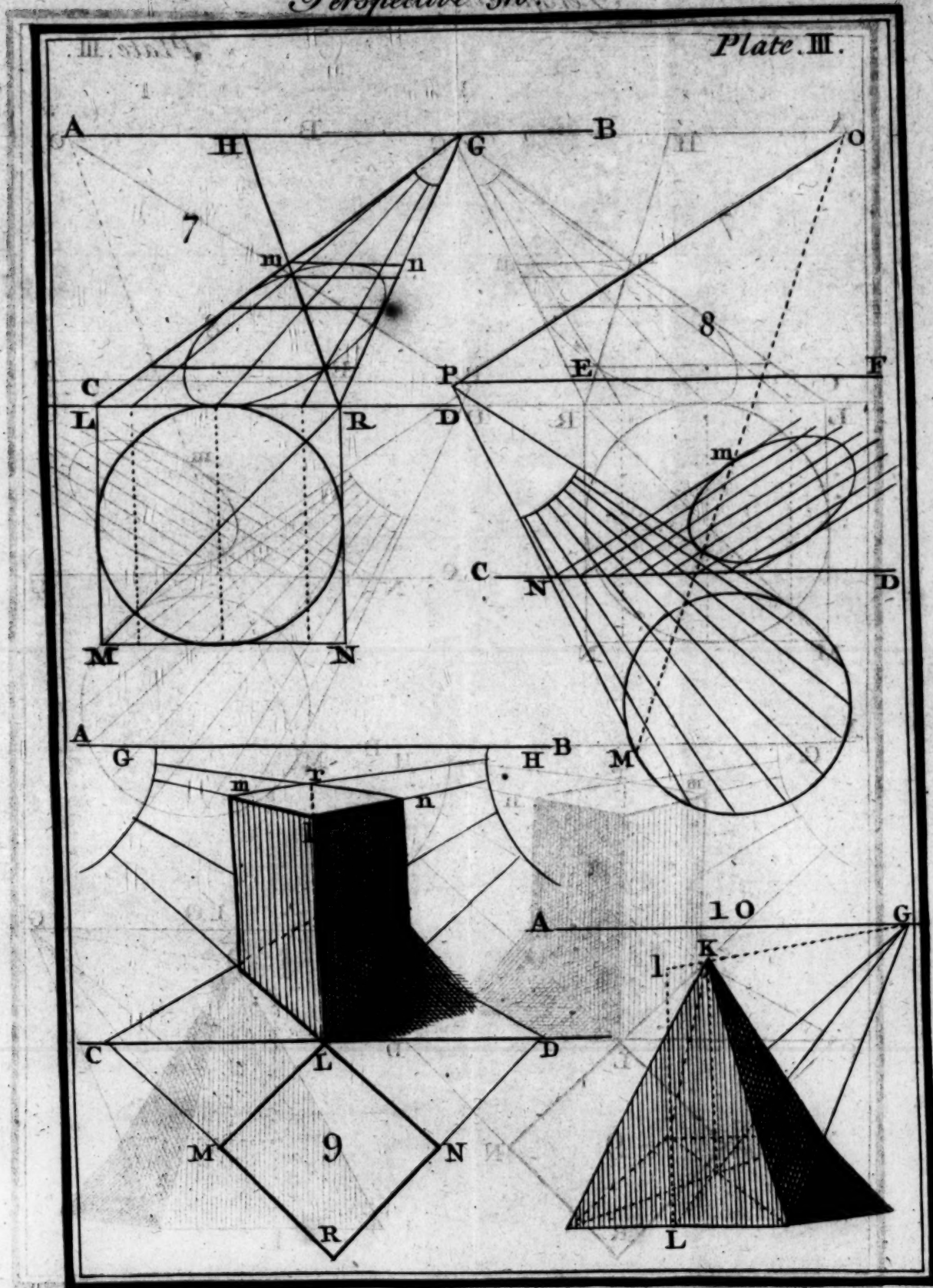
Plate VII. This plate represents the perspective of a house with two wings, seen from a point above the house, so as to see both sides of the roof; DO is the horizontal line,

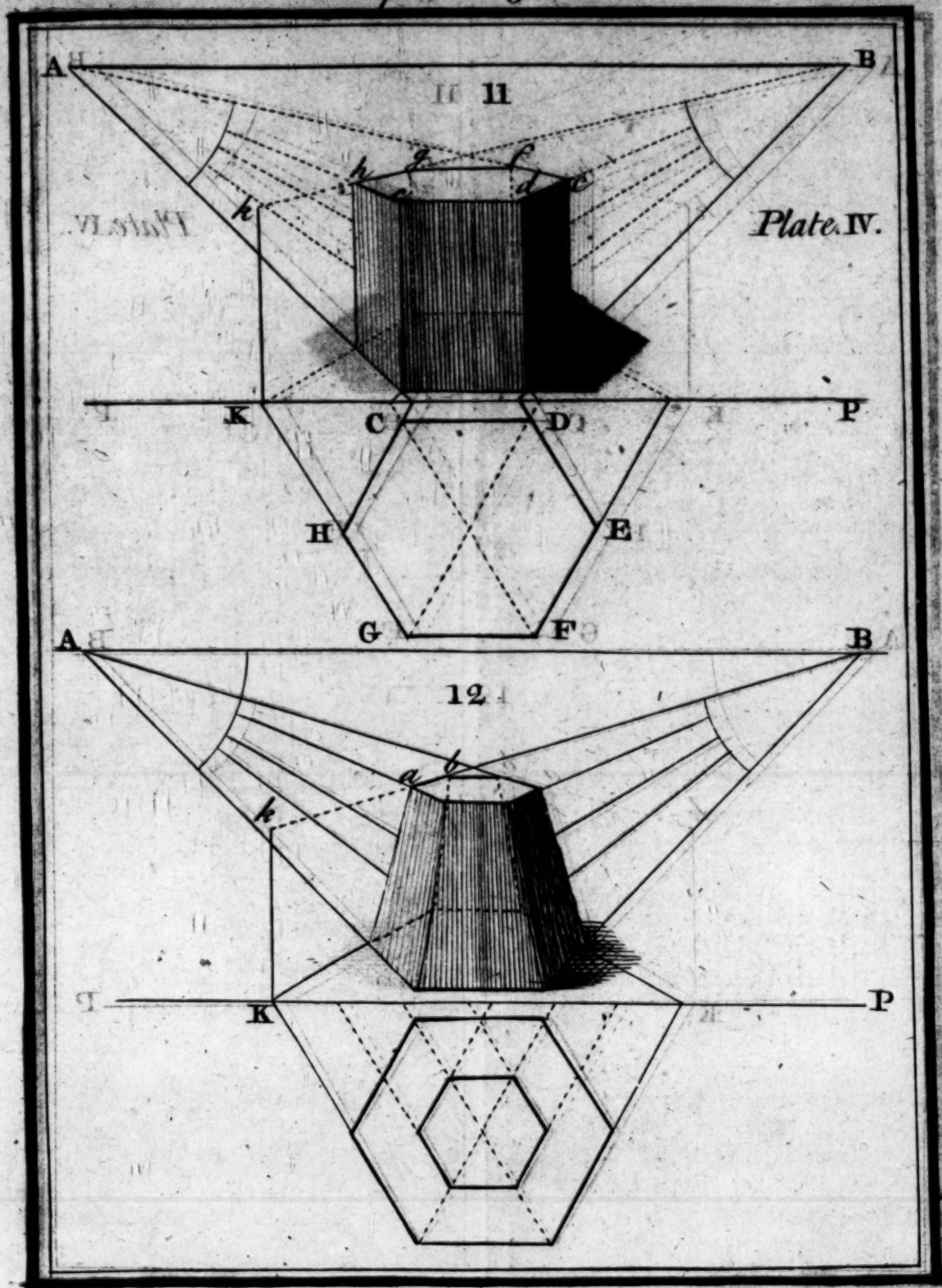
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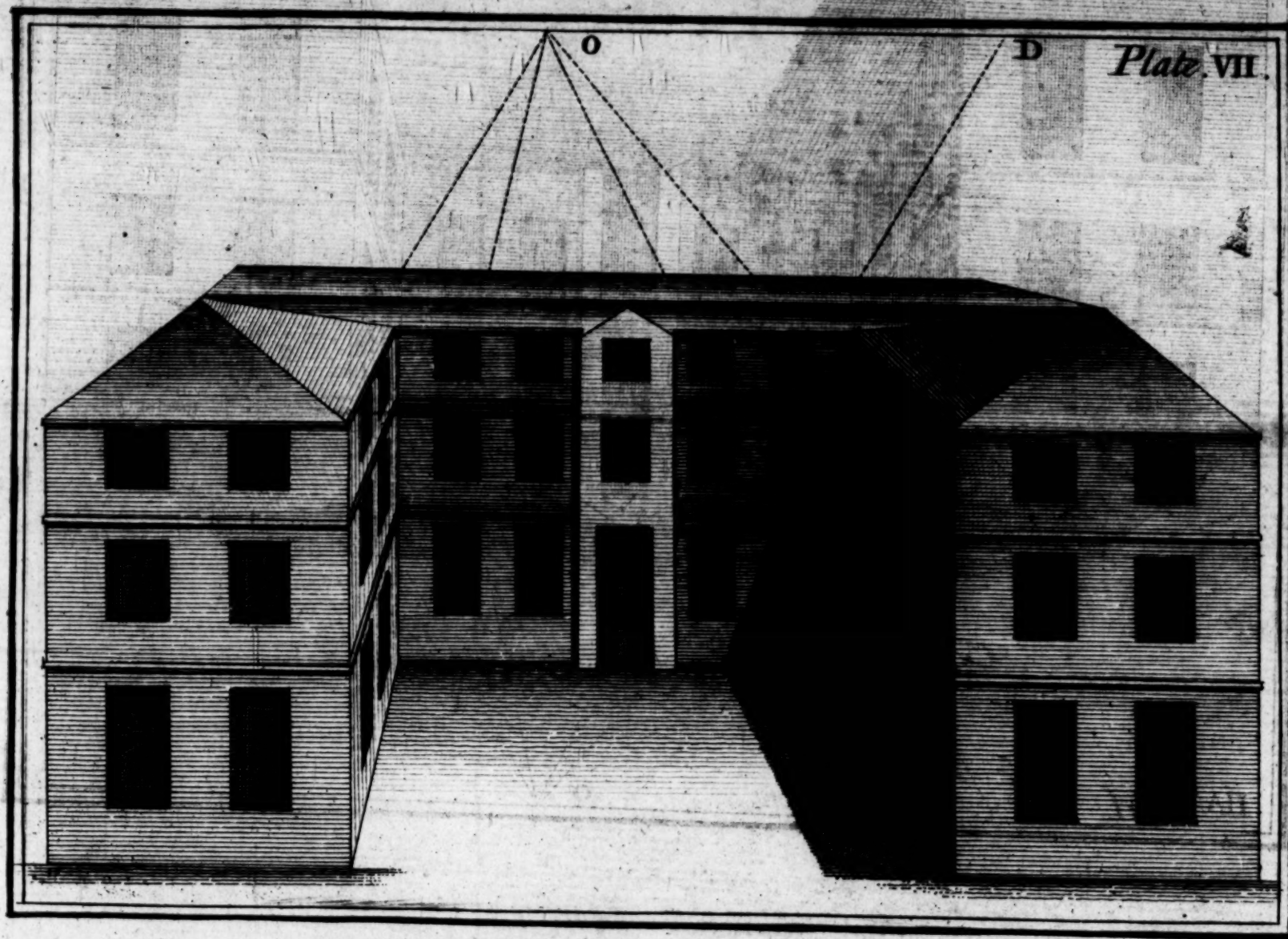
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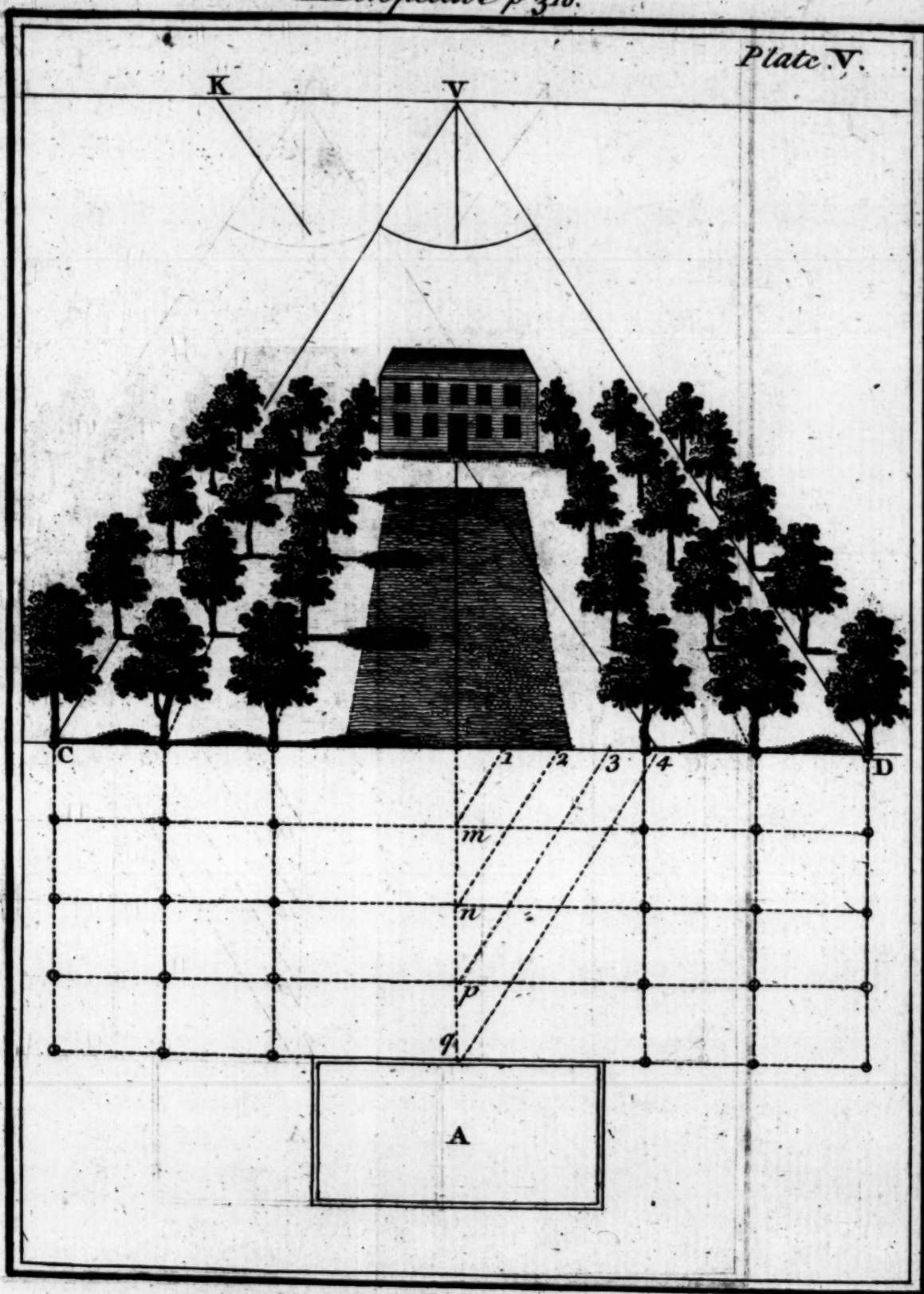
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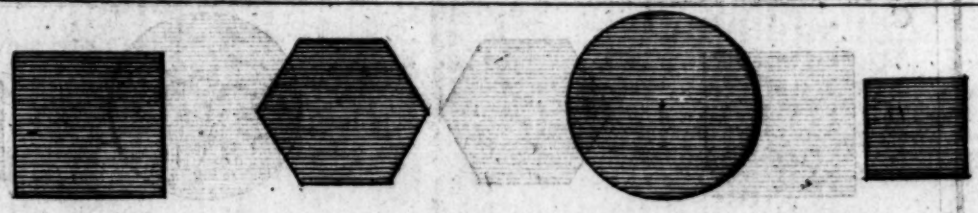
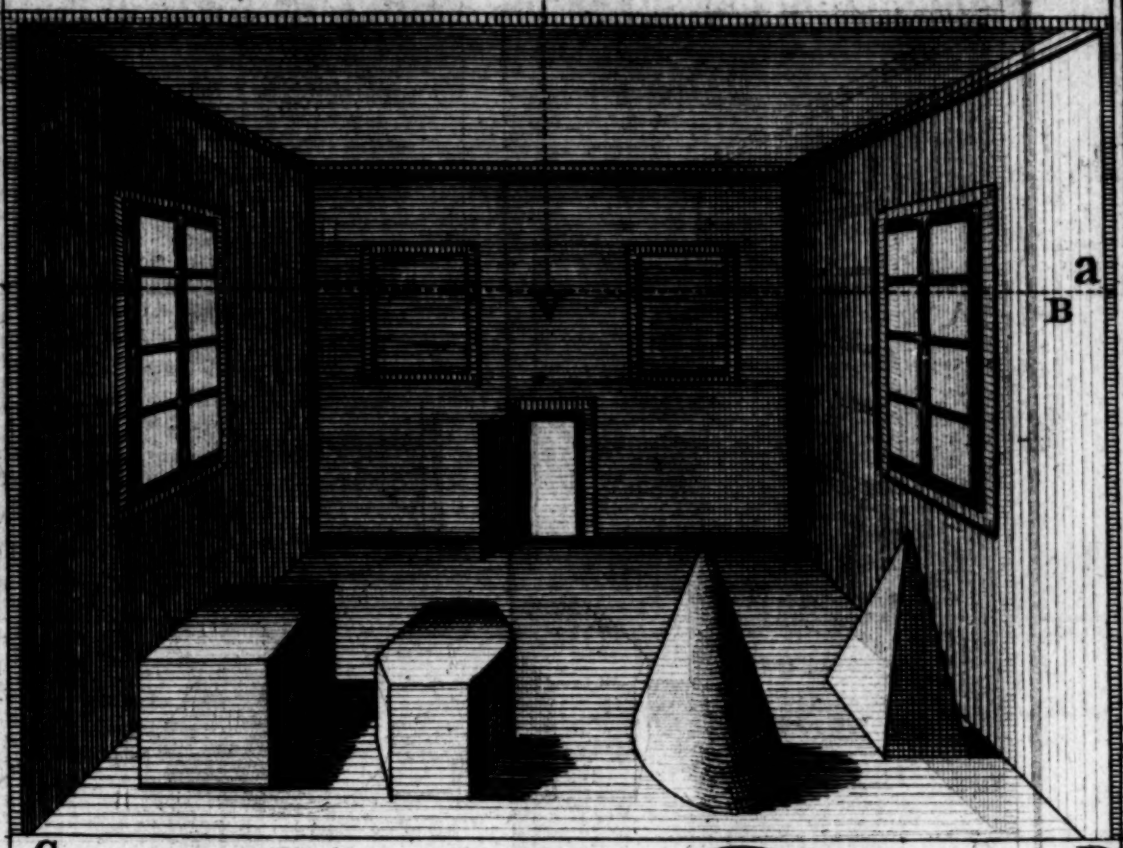


Perspective. p. 310.



Perspective p. 310.

Plate VI.



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line, O the vanishing point of the perpendiculars to the ground plane, DO the distance of the eye from the horizontal line. We shall not give any particulars concerning this perspective, as differing in nothing else from the former, than that the horizontal line passes thro' the picture, whereas it is above it here.

These are all the rules necessary in common perspective, and which are spun out into large volumes by other authors, without the least improvement; for none of them have shewn how to find the perspectives upon concave or convex surfaces, nor upon a figure of several faces, much less, by reflection or refraction; so that all that has been wrote upon perspective by the most voluminous authors, is no more than what is contained in our first problem, examples excepted.

Of LIGHT and SHADOWS.

It is not sufficient to draw well; Plans, Elevations and Perspectives, without they are properly shaded, will not represent the object in an agreeable and natural manner. The light is supposed in general to come from the left side; in plans, the sides to the left of hills, trees and other high objects, is kept as light as can be, and the opposite side is shaded dark; on the contrary, the left side of hollows, and banks of ditches, rivers, is shaded dark, and the opposite side light. In elevations and perspectives, the same rules are observed, and the shadows which the picture casts on the ground, are generally determined by parallel lines drawn from the angles of the shady sides, to the ground plane, excepting when the light is supposed to come from a candle, which is seldom done. The shades or colours of figures must diminish in proportion as they recede from the fore ground; for as objects far off appear less distinct, and their natural colour diminishes in proportion to the distance, and the quantity of air between the eye and the object: If the keeping, as it is called, is not rightly preserved, the picture will never have the desired effect; many printed landscapes are defective in this respect, altho' very well drawn and executed.

CHOICE

CHOICE of DISTANCES.

When the eye is too near the picture, the hind parts appear too much contracted in regard to the fore one, and the perspective appears distorted; and when the eye is too far from the object, the parts become too small and indistinct; as it appears to a spectator, when he stands too near a building he cannot see all the parts without turning the eye round, contrary to the rules of perspective; and when he stands too far off, he cannot see any part distinctly.

If two lines are drawn from the spectator's eye to the extremity of a building, the angle made by these lines in the spectator's eye should never be less than 60 degrees, nor greater than 90, according to the painters; but in trees or figures it chiefly depends on the draughtman's fancy. When the perspective of a building is drawn from its plan, the plan must be reversed, otherwise the perspective will be so; since by looking at a building, the right side is at the left, and the left at the right of the spectator.

F I N I S.

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